Intra-beam Scattering

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Outline

- Review: Deterministic versus statistical approach to particle dynamics
- Liouville’s theorem and “non-Liouvillean” effects
- Moment analysis of the Vlasov-Fokker-Planck equation
- Determination of the Fokker-Planck coefficients (sketch)
- Generalized beam envelope equations
- Emittance growth rates
- Numerical examples, comparison with measurements
- Conclusions
- Appendix: Remarks on irreversibility in computer simulations of beams

Review: Deterministic versus Statistical Approach

Intra-beam scattering: multiple small-angle Coulomb scattering within a charged particle beam.

- $\rightsquigarrow$ Mechanism that limits the life time of beams in storage rings.
- $\rightsquigarrow$ N-body problem with N very large, which is fully determined by both the coupled set of single particle equations of motion

$$m\frac{d^2}{dt^2}\boldsymbol{x}_i - \boldsymbol{F}_{\text{ext}}(\boldsymbol{x}_i, t) - \frac{q^2}{4\pi\epsilon_o}\sum_{j\neq i}\frac{\boldsymbol{x}_i - \boldsymbol{x}_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|^3} = 0\,, \quad i = 1, \ldots, N$$

and the initial condition

$$\big(\boldsymbol{x}_1(t_0), \boldsymbol{v}_1(t_0), \ldots, \boldsymbol{x}_N(t_0), \boldsymbol{v}_N(t_0)\big)\,.$$

- $\rightsquigarrow$ The particles actually *do* interact with each other $\Rightarrow$ collective effect
- $\rightsquigarrow$ The effect of intra-beam scattering originates in the granular nature of the beam's charge distribution.

Liouville's theorem (proper) states that the volume element $d\Gamma$

$$d\Gamma = d\boldsymbol{x}_1 d\boldsymbol{v}_1 \dots d\boldsymbol{x}_N d\boldsymbol{v}_N = d\boldsymbol{x}'_1 d\boldsymbol{v}'_1 \dots d\boldsymbol{x}'_N d\boldsymbol{v}'_N$$

of a Hamiltonian N particle system is *invariant* with respect to *canonical transformations* — in particular with respect to the system's time evolution.

⇝ just a one invariant out of the set of Poincaré invariants.

We may easily proof Liouville's theorem by showing that the determinant of the Jacobi matrix associated with a canonical transformation is unity.

Note: the projection of $d\Gamma$ to subspaces — especially to the 6-dimensional phase-space or 2-dimensional phase-space planes — does *not* necessarily provide a conserved quantity!

We summarize:

⇝ The deterministic approach "works" but is by far too costly.

⇝ Alternative: description in terms of statistical mechanics:
continuous description of beam dynamics.

For the continuous description of the ion beam, we define a 6-dimensional phase-space probability density function

$$f = f(\boldsymbol{x}, \boldsymbol{v}, t)$$

- $\leadsto$ $f\, d\boldsymbol{x}\, d\boldsymbol{v}$ provides the probability of finding a particle inside the volume $d\boldsymbol{x}\, d\boldsymbol{v}$ around the phase-space point $(\boldsymbol{x}, \boldsymbol{v})$ at time t.
- $\leadsto$ f is a smooth function of the phase-space variables.
- $\leadsto$ f does *not* provide information on individual particles.

The "special" Liouville theorem

$$\frac{df}{dt} = 0$$

applies for systems with non-negligible Coulomb interaction only if

1. no particle losses $\Longleftrightarrow \int f\, d\boldsymbol{x}\, d\boldsymbol{v} = \text{const.}$
2. the system can be described to sufficient accuracy by a *continuous* Hamilton function. This means that we must be allowed to treat the internal space charge forces analogously to an external force field.

We observe:

The statistical description smoothes out the effects reflecting the actual charge granularity.

$\leadsto$ The usual equation of motion of the phase-space probability density in the form given by Liouville's theorem $df/dt = 0$ does *not* cover intra-beam scattering effects.

“Non Liouvillean” Effects

$\rightsquigarrow$ An additional equation, hence a generalization of the “special” Liouville theorem is necessary (Chandrasekhar (1943)):

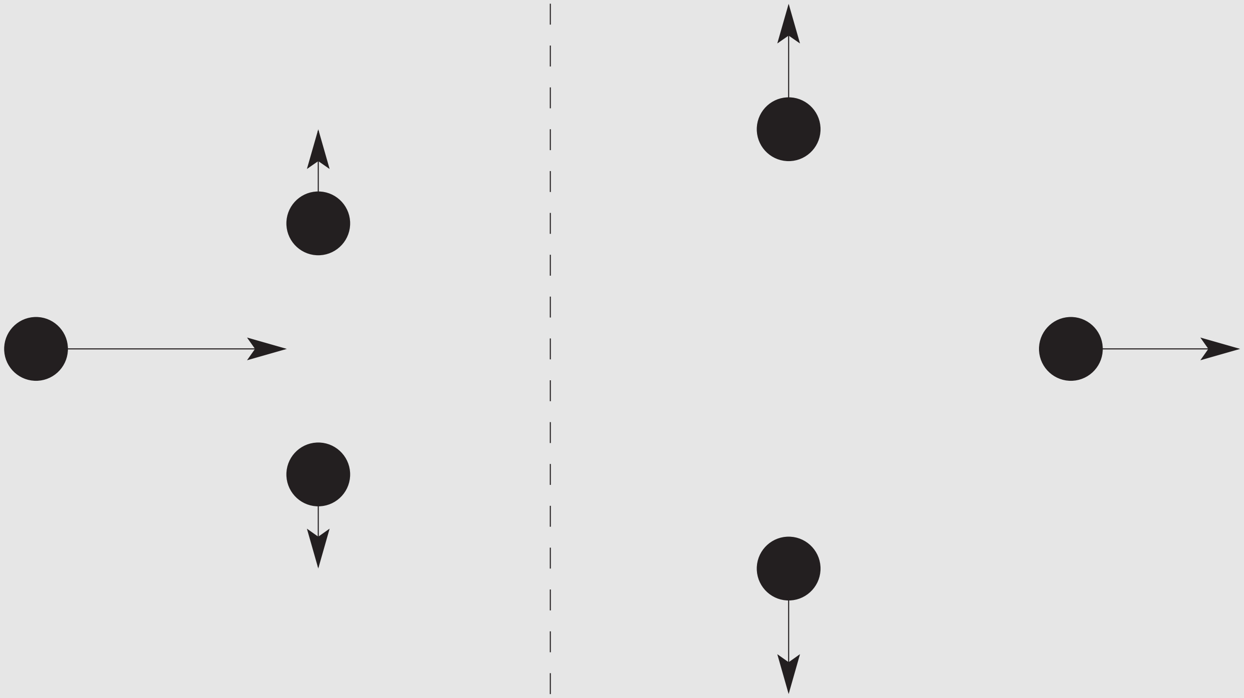
$$\frac{df}{dt} = 0 + \left[\frac{\partial f}{\partial t}\right]_{\mathrm{NL}} .$$

If the non-Liouvillean effects are small compared to the macroscopic forces (smooth space charge and external forces), we can describe them by the Fokker-Planck equation:

$$\left[\frac{\partial f}{\partial t}\right]_{\mathrm{NL}} = -\sum_i \frac{\partial}{\partial v_i}\left\{F_i(\boldsymbol{v},t)\, f\right\} + \sum_{i,j} \frac{\partial^2}{\partial v_i \partial v_j}\left\{D_{ij}(\boldsymbol{v},t)\, f\right\}$$

D_{ij} are referred to as elements of the “diffusion tensor” and the F_i as elements of the “drift vector” that describes the “dynamical friction forces”. They must be determined appropriately depending on the nature of the underlying physical process.

The effect of “dynamical friction” for repelling forces may be visualized as



before and after closest encounter

As is easily verified, a friction also occurs for attracting forces.

We summarize:

- The statistical description replaces the original problem of solving $6N$ coupled first-order equations by *one* equation of motion for the 6-dimensional probability density $f(\boldsymbol{x}, \boldsymbol{v}, t)$.
- We give up the knowledge on individual particles $\Rightarrow$ simplification.
- The intra-beam scattering effects are modeled by a diffusion process in velocity space that is opposed by a dynamical friction force. This process evolves within an effective potential given by the external focusing and the smooth part of the self-fields.
- Friction and diffusion effects depend on each other $\rightsquigarrow$ fluctuation-dissipation theorem.
- Remark: The effect of irreversibility occurs as a consequence of this description.

Explicitly, the generalized Liouville equation for $f(\boldsymbol{x}, \boldsymbol{v}, t)$ writes

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} + \sum_{i=1}^{3} \ddot{x}_i \frac{\partial f}{\partial v_i} = \left[\frac{\partial f}{\partial t}\right]_{\mathrm{NL}} ,$$

which — after inserting the equations of motion — is usually referred to as the "Vlasov" equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f + \frac{1}{m} \left(\vec{F}_{\mathrm{ext}} + q \vec{E}_{\mathrm{ssc}} \right) \cdot \vec{\nabla}_v f = \left[\frac{\partial f}{\partial t}\right]_{\mathrm{NL}} .$$

Herein $\vec{F}_{\mathrm{ext}}$ stands for the external forces applied to the beam by focusing elements (such as quadrupoles) and $\vec{E}_{\mathrm{ssc}}$ for the smooth part of the space charge forces. $\vec{E}_{\mathrm{ssc}}$ depends on f via Poisson's equation:

$$\mathrm{div}\, \vec{E}_{\mathrm{ssc}}(\boldsymbol{x}, t) = \frac{q}{\epsilon_0} \int f(\boldsymbol{x}, \boldsymbol{v}, t) \, d\boldsymbol{v} .$$

$[\partial f / \partial t]_{\mathrm{NL}}$ denotes the Fokker-Planck terms.

We observe:

- The coupled set of Vlasov, Poisson, and Fokker-Planck equations provides the equation of motion for f that includes intra-beam scattering effects.
- Even this equation is too costly to solve directly.
- We do not really want to know $f(\boldsymbol{x}, \boldsymbol{v}, t)$ in detail.
- We must switch to even more global quantities to efficiently estimate intra-beam scattering effects.

Moment analysis of the Vlasov-Fokker-Planck equation

The usual way to switch to even more global physical quantities is to consider "moments" of $f(\boldsymbol{x},\boldsymbol{v},t)$ (F. Sacherer, P. Lapostolle (1970)):

$$\langle x_i^2 \rangle = \int x_i^2 \, f(\boldsymbol{x},\boldsymbol{v},t) \, d\boldsymbol{x} d\boldsymbol{v} \, , \quad i = 1,2,3 \, .$$

$\sqrt{\langle x_i^2 \rangle}$ is proportional to the actual beam width in x_i.

Idea: Instead of solving the equation of motion for f, we are going to set up and solve the equations of motion for the second beam moments.

The derivatives of the moments are calculated according to

$$\frac{d}{dt} \langle x_i^2 \rangle = \int x_i^2 \, \frac{\partial f}{\partial t} \, d\boldsymbol{x} d\boldsymbol{v} \, ,$$

inserting $\partial f / \partial t$ from the Vlasov-Fokker-Planck equation.

Integrating by parts, we obtain for each phase-space plane i a coupled set of "moment" equations

$$\frac{d}{dt}\left\langle x_i^2\right\rangle - 2\left\langle x_i v_i\right\rangle = 0$$

$$m\frac{d}{dt}\left\langle x_i v_i\right\rangle - m\left\langle v_i^2\right\rangle - \left\langle x_i F_{\mathrm{ext},i}\right\rangle - q\left\langle x_i E_{\mathrm{ssc},i}\right\rangle = \left\langle x_i F_{\mathrm{fr},i}\right\rangle$$

$$m\frac{d}{dt}\left\langle v_i^2\right\rangle - 2\left\langle v_i F_{\mathrm{ext},i}\right\rangle - 2q\left\langle v_i E_{\mathrm{ssc},i}\right\rangle = 2\left\langle v_i F_{\mathrm{fr},i}\right\rangle + 2m\left\langle D_{ii}\right\rangle$$

As usual, we define the rms emittance $\varepsilon_i(t)$ as

$$\varepsilon_i^2(t) = \left\langle x_i^2\right\rangle\left\langle v_i^2\right\rangle - \left\langle x_i v_i\right\rangle^2$$

The time derivative of the rms emittance may be arranged as

$$\frac{d}{dt}\varepsilon_i^2(t) = \left.\frac{d}{dt}\varepsilon_i^2(t)\right|_{\mathrm{ext}} + \left.\frac{d}{dt}\varepsilon_i^2(t)\right|_{\mathrm{ssc}} + \left.\frac{d}{dt}\varepsilon_i^2(t)\right|_{\mathrm{fp}}$$

$\frac{d}{dt}\varepsilon_i^2(t)\Big|_{\text{ext}}$ and $\frac{d}{dt}\varepsilon_i^2(t)\Big|_{\text{ssc}}$ describe the emittance change due to non-linear external focusing forces and smooth non-linear electric self-fields.

$$\begin{aligned}\frac{m}{2}\frac{d}{dt}\varepsilon_i^2(t)\Bigg|_{\text{ext}} &= \left\langle x_i^2\right\rangle \left\langle v_i F_{\text{ext},i}\right\rangle - \left\langle x_i v_i\right\rangle \left\langle x_i F_{\text{ext},i}\right\rangle \\ &= 0 \quad \Longleftrightarrow \quad F_{\text{ext},i} \propto x_i\,.\end{aligned}$$

$$\frac{m}{2}\frac{d}{dt}\varepsilon_i^2(t)\Bigg|_{\text{ssc}} = q\left[\left\langle x_i^2\right\rangle \left\langle v_i E_{\text{ssc},i}\right\rangle - \left\langle x_i v_i\right\rangle \left\langle x_i E_{\text{ssc},i}\right\rangle\right]\,.$$

The third contribution to the change of the emittance emerges from the terms of Fokker-Planck equation

$$\frac{m}{2}\frac{d}{dt}\varepsilon_i^2(t)\Bigg|_{\text{fp}} = \left\langle x_i^2\right\rangle \left\langle v_i F_{\text{fr},i}\right\rangle - \left\langle x_i v_i\right\rangle \left\langle x_i F_{\text{fr},i}\right\rangle + m\left\langle x_i^2\right\rangle \left\langle D_{ii}\right\rangle\,.$$

$\leadsto$ the emittance growth depends on both the Fokker-Planck coefficients *and* and the specific shape of the envelope functions.

Determination of the Fokker-Planck coefficients

Up to now, the physical nature of the diffusion/friction process has not been specified.

For the effect of intra-beam scattering, $F_{\mathrm{fr},i}$ can be obtained by

1. solving the equations of motion for a single small angle scattering event
2. averaging over all allowable impact parameters,
3. subsequently averaging over the velocity distribution of the beam.

Not too far from thermodynamic equilibrium $F_{\mathrm{fr},i}$ is given by (Chandrasekhar (1943), Jansen (1990), and others):

$$F_{\mathrm{fr},i} = -m\beta_f v_i \,, \qquad \beta_f = \frac{16\sqrt{\pi}}{3}\frac{Z^4}{A^2} \cdot n c r_c^2 \left(\frac{mc^2}{2kT_{\mathrm{eq}}}\right)^{3/2} \cdot \ln\Lambda$$

with $r_c = e_0^2/(4\pi\epsilon_0 mc^2)$ the classical particle radius, n the particle density, and T_{eq} the equilibrium temperature.

Discussion of the fluctuation-dissipation theorem:

Systems in dynamical equilibrium are governed by

- diffusion: effect that drives a quantity off its steady-state value
- friction (dissipation): effect that causes the decay of this deviation from the steady-state value.

The diffusion process and friction effects are *not* independent of each other.

$\rightsquigarrow$ Both effects are related by a fluctuation-dissipation theorem

$\rightsquigarrow$ Simplest case (isotropic process): Einstein relation

$$D \equiv D_{ii} = \beta_f \frac{k_B T_{\mathrm{eq}}}{m}$$

We will use this simple approximation in our approach.

Generalized beam envelope equations

With

$$F_{\mathrm{fr},i} = -m\beta_f \, v_i \,, \qquad F_{\mathrm{ext},i} = -m\omega_i^2(t)\, x_i$$

we obtain the generalized envelope equation from the first two moment equations

$$\frac{d^2}{dt^2}\sqrt{\langle x_i^2\rangle} + \beta_f \frac{d}{dt}\sqrt{\langle x_i^2\rangle} + \omega_i^2(t)\sqrt{\langle x_i^2\rangle} - \frac{q}{m}\frac{\langle x_i E_{\mathrm{ssc},i}\rangle}{\sqrt{\langle x_i^2\rangle}} - \frac{\varepsilon_i^2(t)}{\sqrt{\langle x_i^2\rangle}^{\,3}} = 0$$

For the Fokker-Planck emittance change, the above approximations lead to

$$\frac{1}{\langle x_i^2\rangle}\frac{d}{dt}\varepsilon_i^2(t)\bigg|_{\mathrm{fp}} = 2\beta_f\left(\frac{k_B T_{\mathrm{eq}}}{m} - \frac{\varepsilon_i^2(t)}{\langle x_i^2\rangle}\right)$$

$\rightsquigarrow$ simple temperature relaxation equation.

$\rightsquigarrow$ closed set of differential equations for $\sqrt{\langle x_i^2\rangle}$ and $\varepsilon_i^2(t)$.

Beam “Temperatures”

Non-equilibrium temperature:

For charged particle beams, we define the generalized, non-equilibrium “temperature” $k_B T_i$ as the instantaneous *incoherent* part of the kinetic energy of the beam particles in the i-th degree of freedom:

$$k_B T_i \equiv m \left\langle \left(v_i^{\mathrm{inc}} \right)^2 \right\rangle , \qquad v_i^{\mathrm{inc}} = v_i - x_i \frac{\langle x_i v_i \rangle}{\langle x_i^2 \rangle}$$

since the total kinetic energy $m \left\langle v_i^2 \right\rangle /2$ contains a coherent part if $\langle x_i v_i \rangle \neq 0$. With the rms emittance ε_i defined by

$$\varepsilon_i^2(t) = \left\langle x_i^2 \right\rangle \left\langle v_i^2 \right\rangle - \left\langle x_i v_i \right\rangle^2$$

the non-equilibrium “temperature” $k_B T_i$ of the i-th degree of freedom can then be expressed as

$$k_B T_i(t) = m \frac{\varepsilon_i^2(t)}{\langle x_i^2 \rangle}$$

Equilibrium temperature:

For a coasting beam in a strong focusing system, we have

$$T_x > T_{\text{eq}} \Longleftrightarrow T_y < T_{\text{eq}}$$

and vice versa. With $k_B T_{z,b} = m \left\langle (\Delta v_{z,b})^2 \right\rangle$ the longitudinal temperature in the beam frame, we may approximate the equilibrium temperature T_{eq} by

$$\frac{k_B T_{\text{eq}}}{m} = \frac{k_B}{3m}\left(T_x + T_y + T_z\right) = \frac{1}{3}\left(\frac{\varepsilon_x^2}{\langle x^2\rangle} + \frac{\varepsilon_y^2}{\langle y^2\rangle} + \left\langle (\Delta v_{z,b})^2 \right\rangle\right)$$

Emittance growth rates

With the temperature relations, the above formula for the F-P related emittance growth is obtained for the x-direction as

$$\frac{1}{\langle x^2\rangle}\frac{d}{dt}\varepsilon_x^2(t)\bigg|_{\mathrm{fp}} = -\frac{2\beta_f}{3}\left(\frac{2\varepsilon_x^2(t)}{\langle x^2\rangle} - \frac{\varepsilon_y^2(t)}{\langle y^2\rangle} - \left\langle (\Delta v_{z,b})^2 \right\rangle\right) ,$$

or equivalently with the temperature ratios $r_{xy} = T_y(t)/T_x(t)$, $r_{xz} = T_z(t)/T_x(t)$, and $r_{yz} = T_z(t)/T_y(t)$

$$\frac{d}{dt}\ln\varepsilon_x^2(t)\bigg|_{\mathrm{fp}} = \frac{2\beta_f}{3}\left(r_{xy} + r_{xz} - 2\right) ,$$

$\leadsto$ The change of the emittance may be positive as well as negative.

Summing over all three degrees of freedom, we get

$$\frac{d}{dt}\ln\varepsilon_x^2\varepsilon_y^2\varepsilon_z^2\Big|_{\mathrm{fp}} = \frac{2\beta_f}{3}\left(\frac{(1-r_{xy})^2}{r_{xy}} + \frac{(1-r_{xz})^2}{r_{xz}} + \frac{(1-r_{yz})^2}{r_{yz}}\right) \geq 0$$

$\rightsquigarrow$ The total change of all emittances is always positive.

Final result: we integrate this equation over one turn T to obtain the emittance e-folding time τ_{ef} of the total emittance $\varepsilon = \sqrt[3]{\varepsilon_x\varepsilon_y\varepsilon_z}$

$$\tau_{\mathrm{ef}}^{-1} = \tfrac{1}{9}\,\beta_f\,(I_{xy} + I_{xz} + I_{yz})$$

with the “temperature imbalance integrals” I_{xy}, I_{xz}, I_{yz} defined as

$$I_{xy} = \frac{1}{T}\int_0^T \frac{\left[1-r_{xy}(t)\right]^2}{r_{xy}(t)}\,dt \qquad , \qquad r_{xy}(t) = \frac{\varepsilon_y^2}{\langle y^2\rangle}\frac{\langle x^2\rangle}{\varepsilon_x^2}\,.$$

With the abbreviations

$$
\begin{aligned}
a &= \sqrt{\langle x^2\rangle}\,, & b &= \sqrt{\langle y^2\rangle} \\
\delta &= \sqrt{\langle(\Delta p/p)^2\rangle}\,, & D &= \Delta x/(\Delta p/p) \\
k_i^2(s) &= \omega_i(t)/c^2\beta^2\gamma^2\,, & \bar{\varepsilon}_i &= \varepsilon_i/c\beta\gamma \\
\eta &= \gamma^{-2} - D/\rho \\
A &= \sqrt{a^2 + D^2\delta^2} \\
K &= \frac{2Ze_0 I}{4\pi\epsilon_0 mc^3\beta^3\gamma^3} \\
k_f &= \frac{K}{3\sqrt{2\pi}\beta^2}\,\frac{Z^2 r_c}{A}\,\frac{\ln\Lambda}{\bar{\varepsilon}_x\bar{\varepsilon}_y\sqrt{|\eta|}\delta}
\end{aligned}
$$

the complete system of moment equations for a coasting beam with elliptic cross section in real space that propagates through a dispersive system reads:

$$a'' + k_f a' + k_x^2(s)\, a - \frac{K/2}{A(A+b)} a - \frac{\bar{\varepsilon}_x^2}{a^3} = 0$$

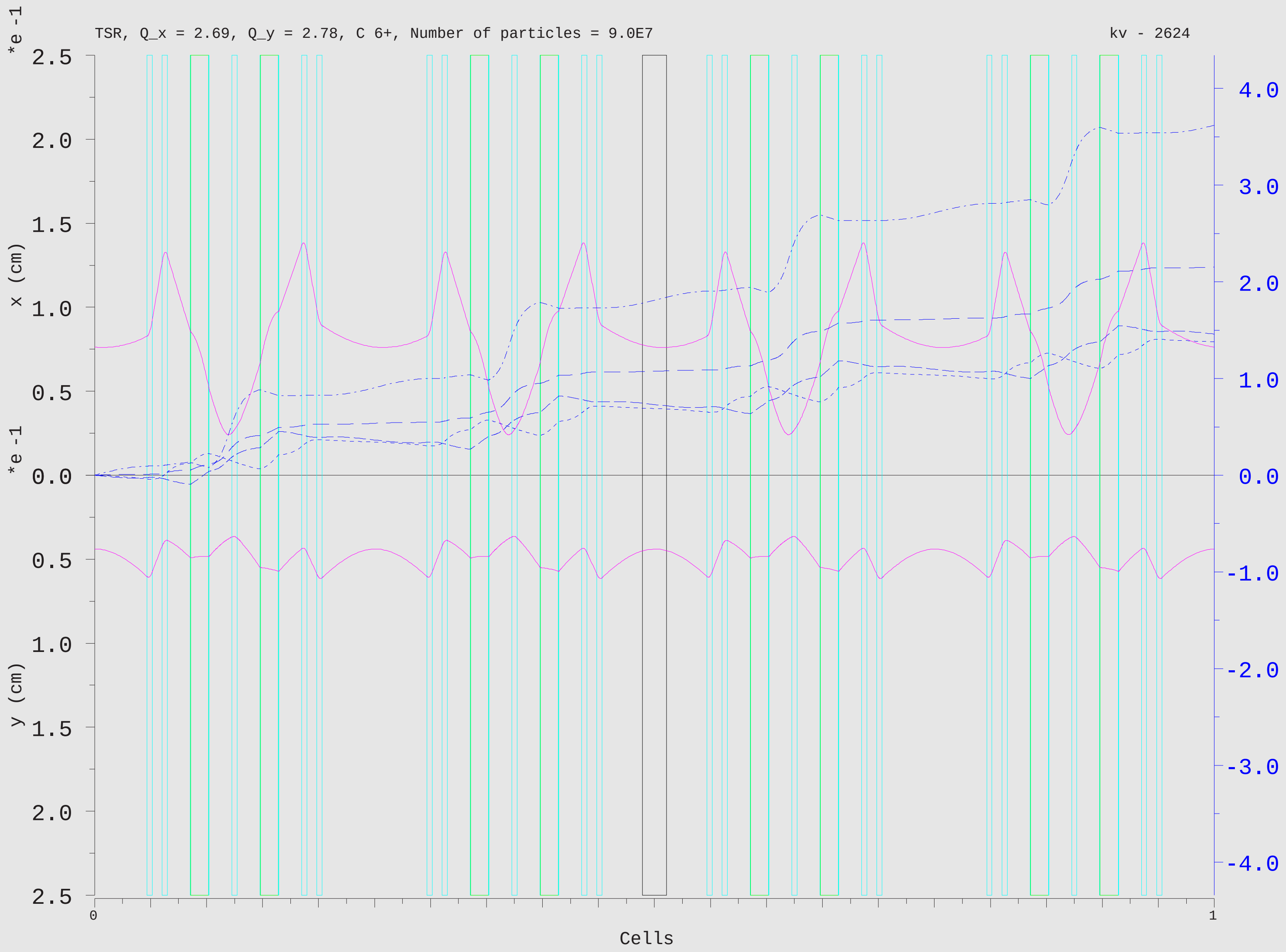
$$b'' + k_f b' + k_y^2(s)\, b - \frac{K/2}{A+b} - \frac{\bar{\varepsilon}_y^2}{b^3} = 0$$

$$D'' + \left(k_x^2(s) - \rho^{-2}(s)\right) D - \frac{K/2}{A(A+b)} D - \frac{1}{\rho} = 0$$

$$\frac{1}{a^2}\frac{d}{ds}\bar{\varepsilon}_x^2 + \tfrac{2}{3} k_f \left(2\frac{\bar{\varepsilon}_x^2}{a^2} - \frac{\bar{\varepsilon}_y^2}{b^2} - |\eta|\delta^2\right) = 0$$

$$\frac{1}{b^2}\frac{d}{ds}\bar{\varepsilon}_y^2 + \tfrac{2}{3} k_f \left(2\frac{\bar{\varepsilon}_y^2}{b^2} - \frac{\bar{\varepsilon}_x^2}{a^2} - |\eta|\delta^2\right) = 0$$

$$|\eta|\frac{d}{ds}\delta^2 + \tfrac{2}{3} k_f \left(2|\eta|\delta^2 - \frac{\bar{\varepsilon}_x^2}{a^2} - \frac{\bar{\varepsilon}_y^2}{b^2}\right) = 0$$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*e -1
x (cm)
*e -1
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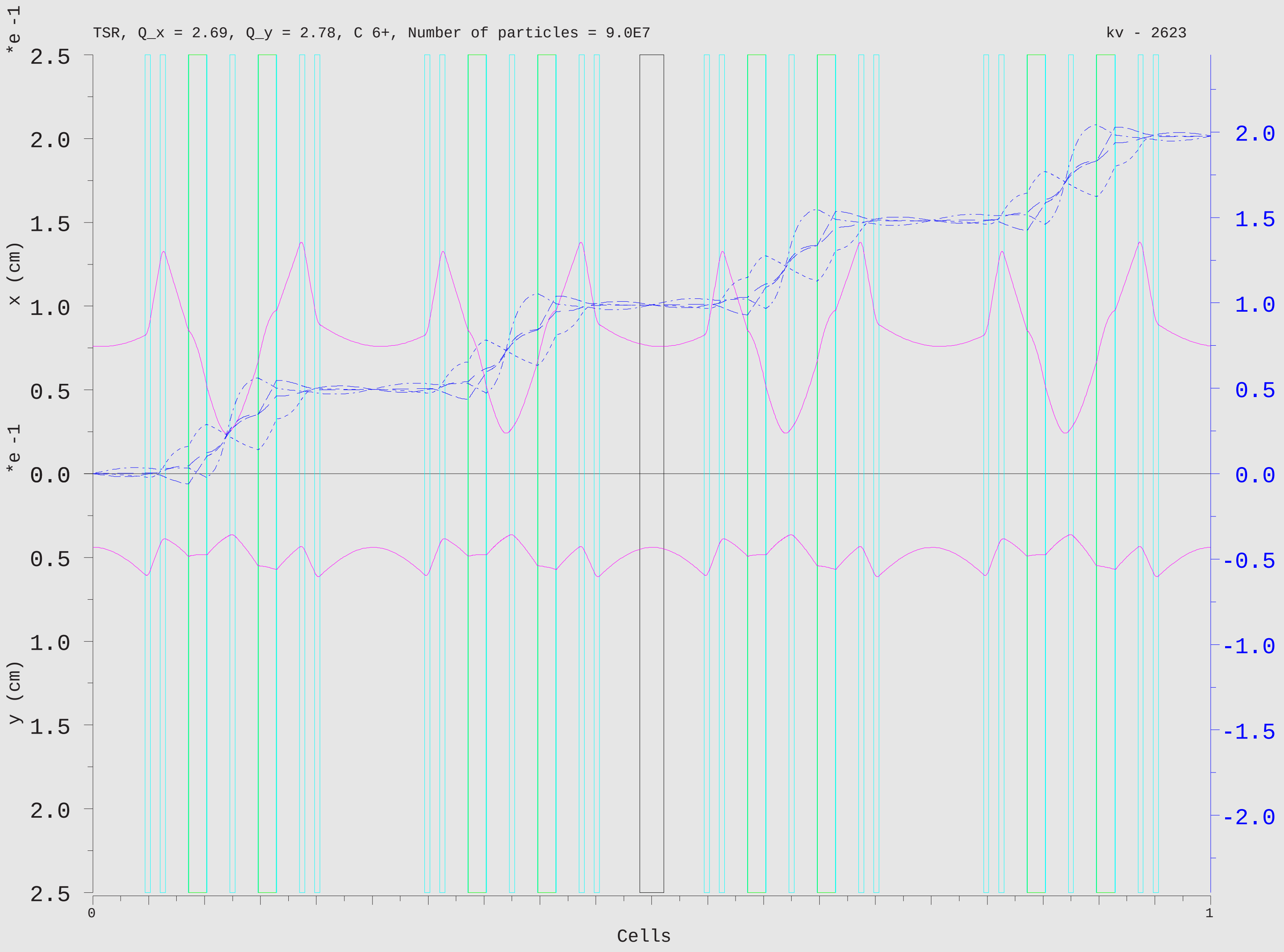
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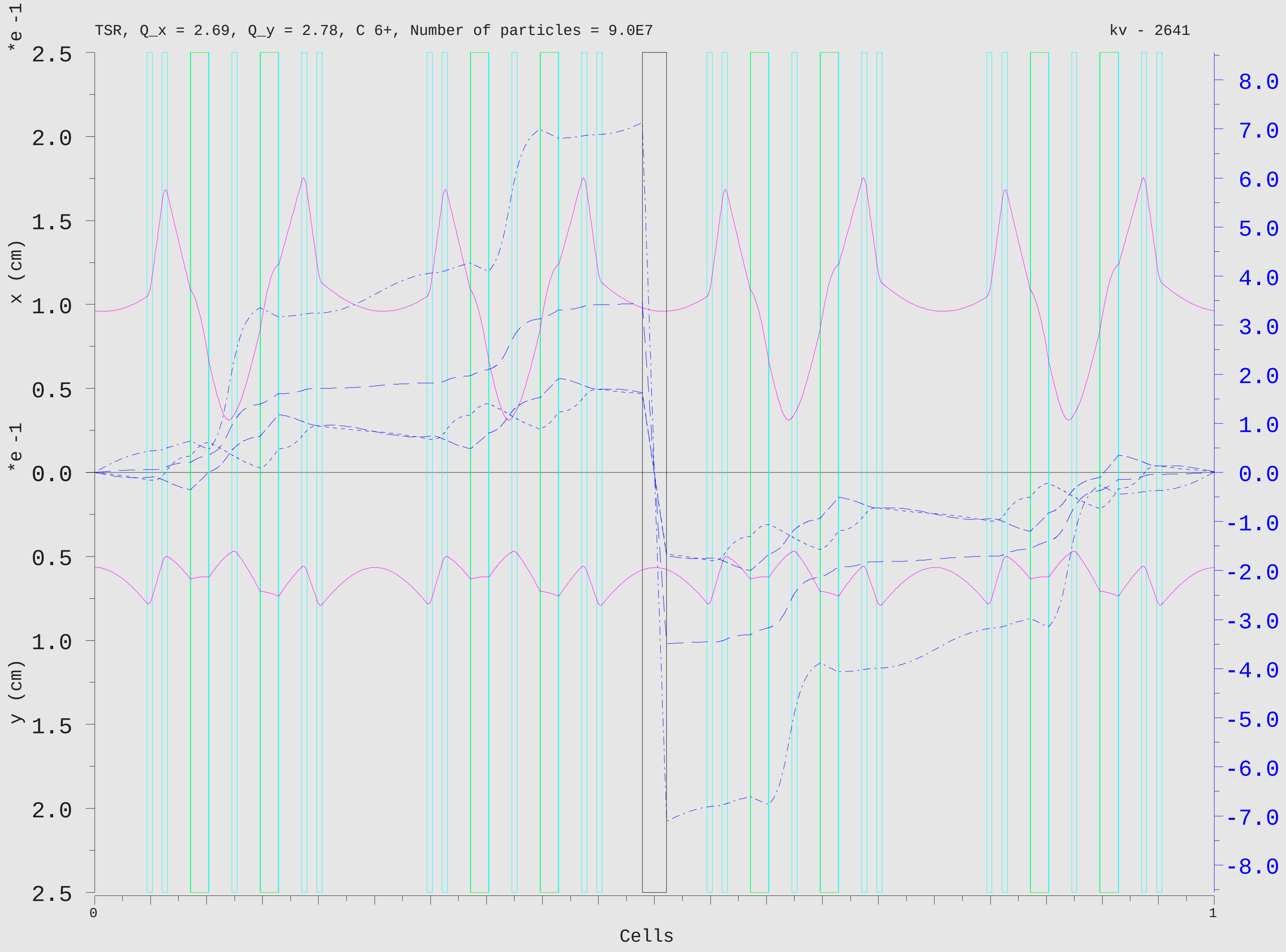
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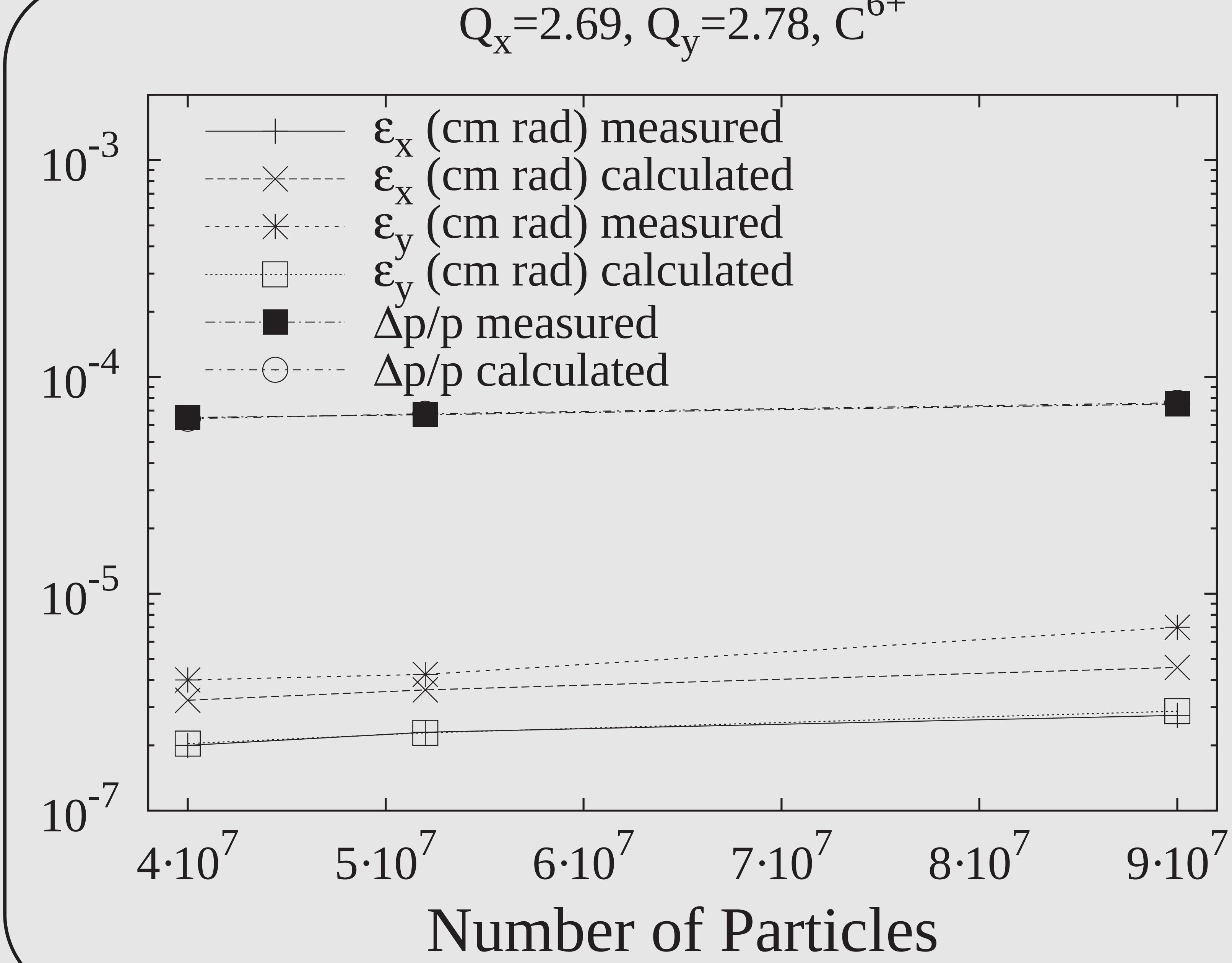
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ε_x (cm rad) calculated
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ε_y (cm rad) calculated
Δp/p measured
Δp/p calculated
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10^{-4}
10^{-5}
10^{-7}
$4\cdot10^7$
$5\cdot10^7$
$6\cdot10^7$
$7\cdot10^7$
$8\cdot10^7$
$9\cdot10^7$
Number of Particles

Conclusions

- The Vlasov-Fokker-Planck equation provides the starting point for analytical approaches in the physics of charged particle beams if the actual charge granularity cannot be neglected.
- The moment analysis of this equation provides a useful extension of Sacherer's moment analysis of the Vlasov equation.
- The emittance growth rates that are due to intra-beam scattering follow straightforwardly from the temperature imbalances and the friction coefficient β_f.
- The determination of the appropriate friction coefficient β_f needs careful examination for each particular case.

Appendix: Remarks on irreversibility

The friction forces $F_{\mathrm{fr},i}$ must always be decelerating.

$$F_{\mathrm{fr},i}(v_i) = -F_{\mathrm{fr},i}(-v_i) \quad , \quad \leadsto D_{ii}(v_i) = D_{ii}(-v_i)\,.$$

Transformation that reverses the direction of time flow:

$$t \longrightarrow -t \qquad \leadsto x_i \longrightarrow x_i, \qquad v_i \longrightarrow -v_i.$$

We may separate the components of the equation of motion for $f(\boldsymbol{x}, \boldsymbol{v}, t)$ with respect to their behavior under time reversal

$$\frac{\partial f}{\partial t} = \boldsymbol{L}_{\mathrm{FP}}\, f\,, \qquad \boldsymbol{L}_{\mathrm{FP}} = \boldsymbol{L}_{\mathrm{rev}} + \boldsymbol{L}_{\mathrm{ir}}\,.$$

The “reversible” operator $\boldsymbol{L}_{\mathrm{rev}}$: terms that change sign under time reversal, hence leave $\partial f/\partial t = \boldsymbol{L}_{\mathrm{rev}}\, f$ invariant.

$\leadsto$ Earlier states are fully restored — just like a movie that is reversed at some instant of time t_0.

The reversible terms agree with the familiar Vlasov equation

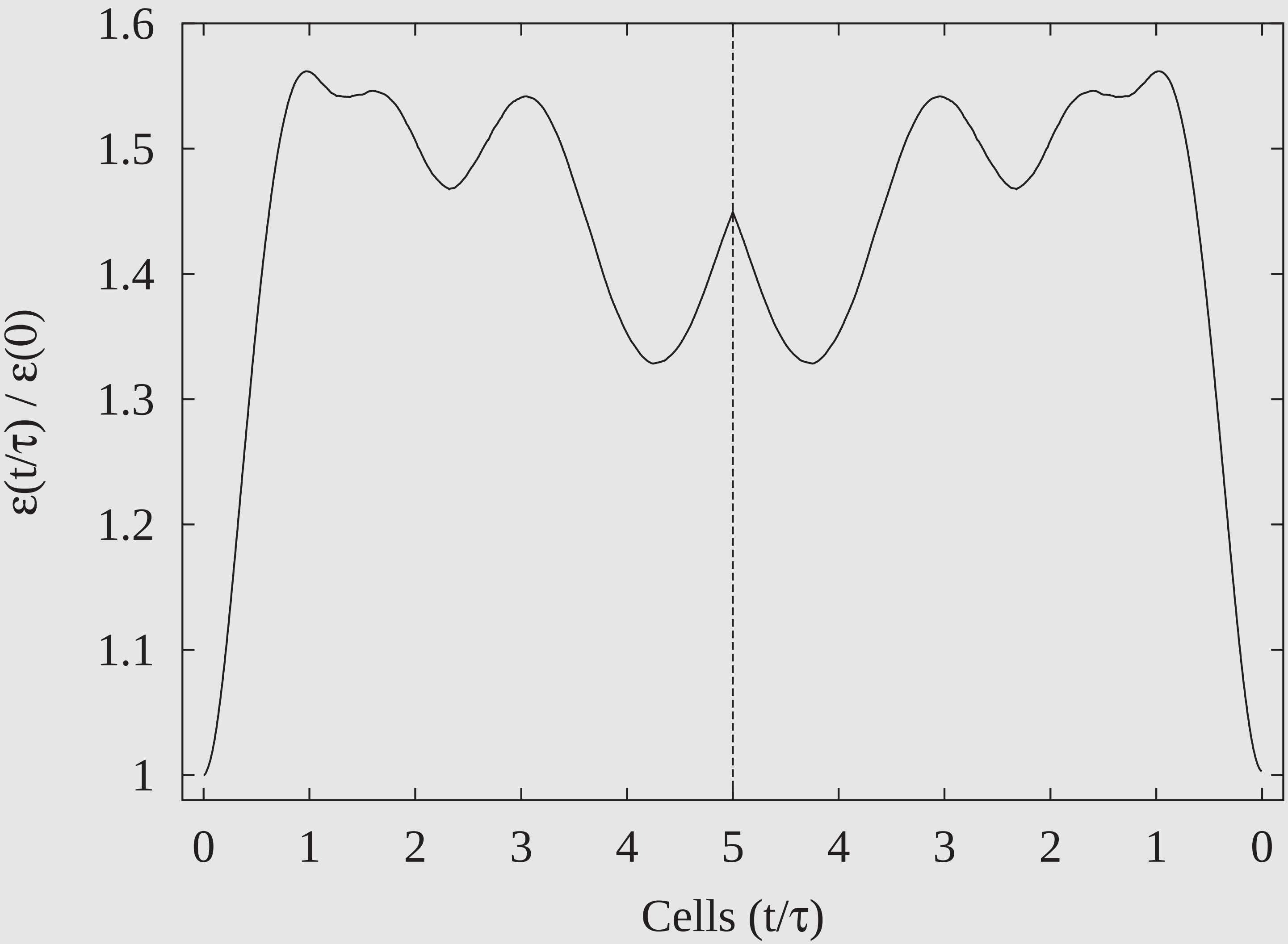
$$\boldsymbol{L}_{\mathrm{rev}} = \sum_{i=1}^{3} \left[-\frac{\partial}{\partial x_i} v_i - \frac{1}{m} \frac{\partial}{\partial v_i} \left(F_{\mathrm{ext},i} + q E_{\mathrm{ssc},i} \right) \right] .$$

The components $\boldsymbol{L}_{\mathrm{ir}}$ that do not change sign are given by the Fokker-Planck terms

$$\boldsymbol{L}_{\mathrm{ir}} = \sum_{i=1}^{3} \frac{\partial}{\partial v_i} \left[-\frac{F_{\mathrm{fr},i}(v_i, t)}{m} + \frac{\partial}{\partial v_i} D_{ii}(v_i, t) \right] .$$

$\boldsymbol{L}_{\mathrm{ir}}$ describes those effects that do *not* depend on the direction of time flow. In other words, it describes the *irreversible* aspects of the particle motion.

Real system: mixture of reversible and irreversible behavior.
Computer simulations of dynamical systems are irreversible even if the coded equations of motion are strictly reversible.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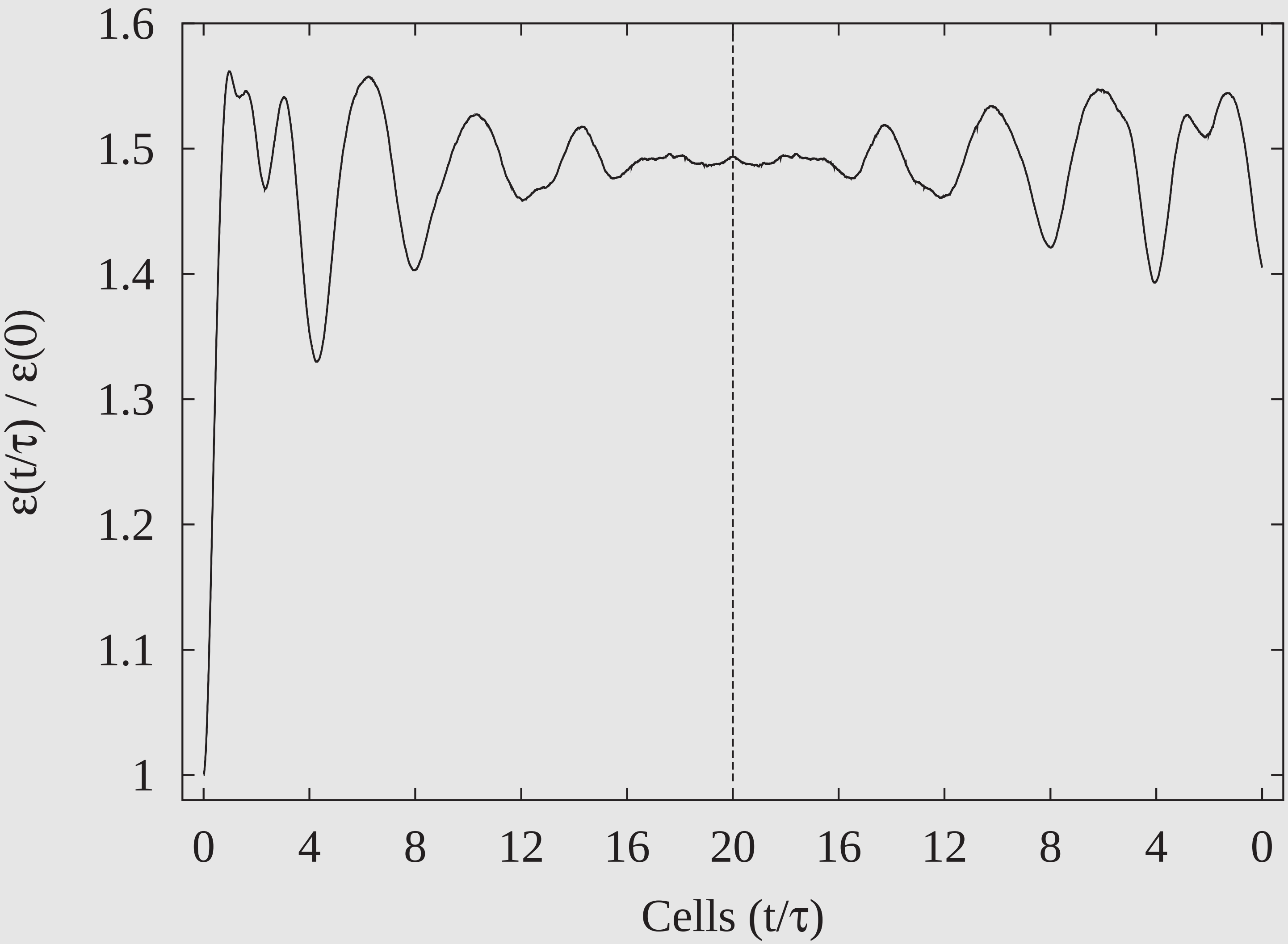
Cells (t/τ)


ε(t/τ) / ε(0)
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1.5
1.4
1.3
1.2
1.1
1
0
4
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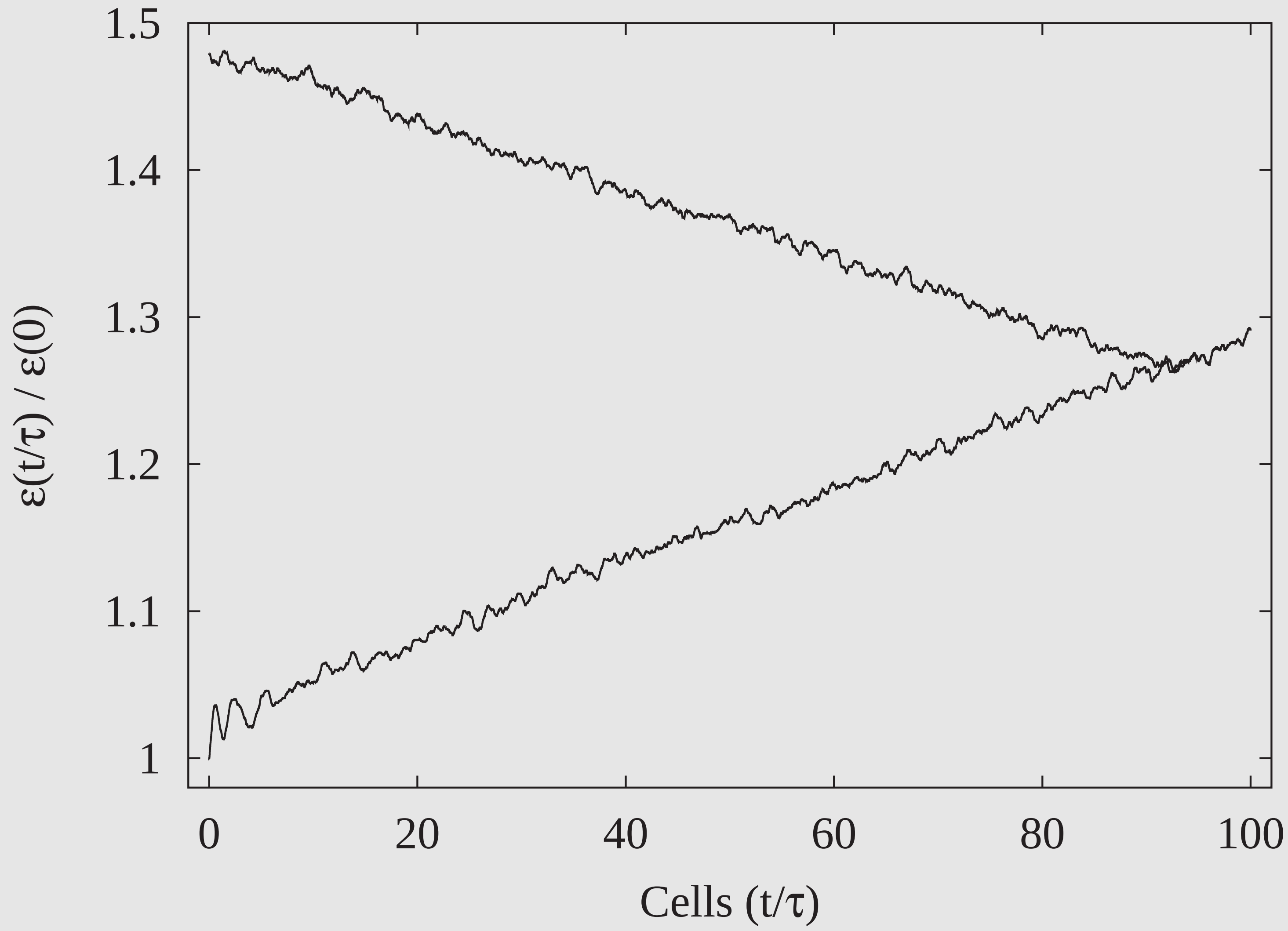
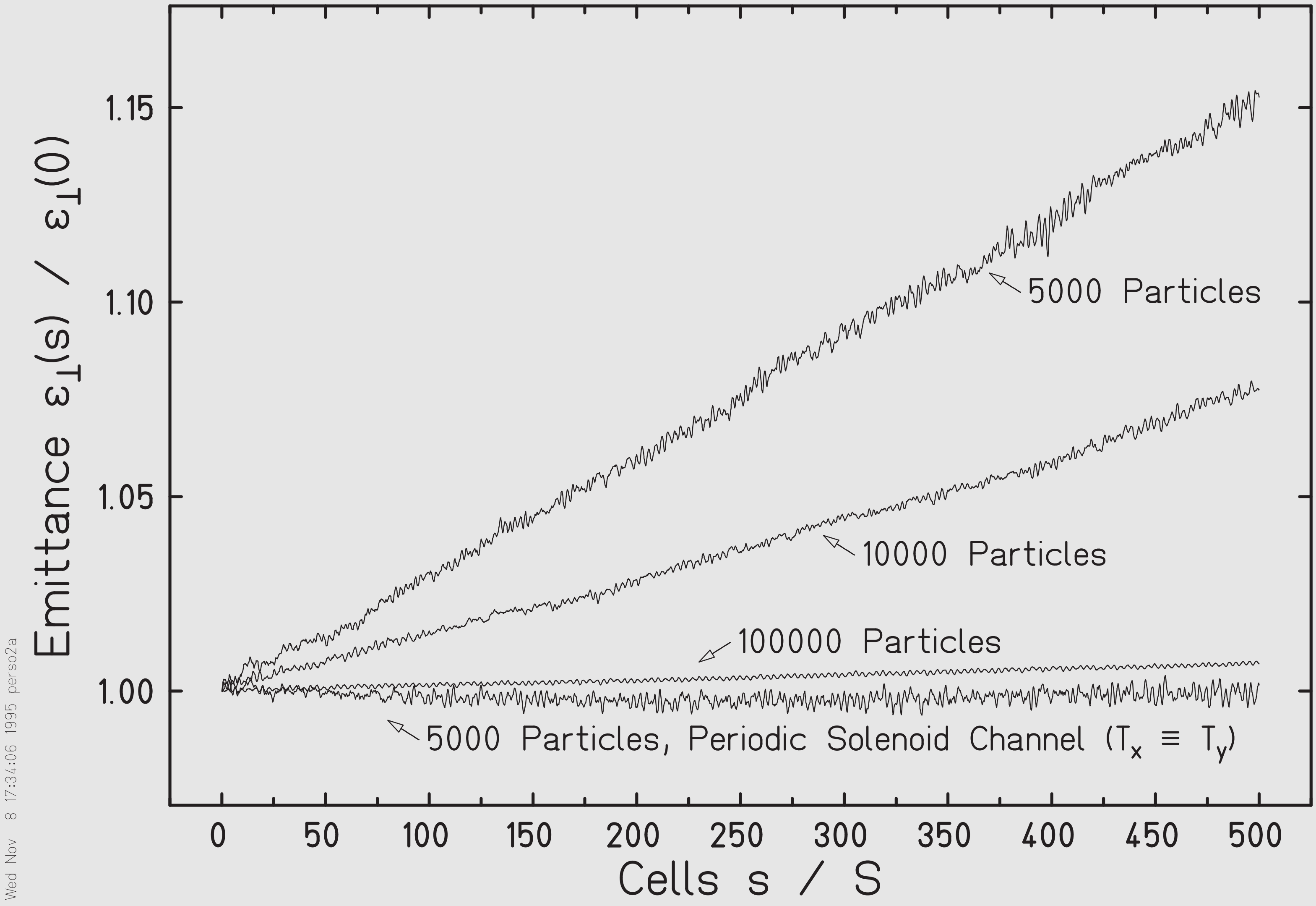
16
20
16
12
8
4
0
Cells (t/τ)


1.5
1.4
1.3
1.2
1.1
1
0
20
40
60
80
100
ε(t/τ) / ε(0)
Cells (t/τ)

Periodic Quadrupole Channel, $\sigma_0 = 60°$, $\sigma = 15°$



The obtained macroscopic emittance growth is determined by both the friction coefficient β_f^{sim} that quantifies the rate of information loss *and* the temperature imbalance I_{xy} along the focusing period S

$$\tau_{\mathrm{ef}}^{-1} = \tfrac{1}{4}\,\beta_f^{\mathrm{sim}} \cdot I_{xy}$$

with I_{xy} defined as

$$I_{xy} = \frac{1}{S}\int\limits_0^S \frac{\left[1 - r_{xy}(s)\right]^2}{r_{xy}(s)}\,ds \qquad , \qquad r_{xy}(s) = \frac{\varepsilon_y^2}{\left\langle y^2\right\rangle}\,\frac{\left\langle x^2\right\rangle}{\varepsilon_x^2}\,.$$

If the number of macro-particles used in the simulation is increased, the system get “smoother”, hence β_f^{sim} becomes smaller whereas I_{xy} is kept constant.
If the system is modified and the number of macro-particles is kept constant, β_f^{sim} remains unchanged whereas I_{xy} is varied.
⇒ If either β_f^{sim} or I_{xy} vanishes, we do no face any FP-related emittance growth.

- Discreteness errors inevitably emerge in computer simulations of dynamical systems.
- The actual time evolution of the simulated system always comprises irreversible aspects — even if the actually coded equations of motion are strictly reversible.
- A computer simulation based on individual particles can never be an *exact* realization of a solution of the Vlasov equation.
- The results of the computer simulation may be regarded *equivalently* as an exact solution of a Vlasov-Fokker-Planck equation.
- The crucial point for the correct interpretation of computer simulations is to keep in mind that the noise-related emittance growth depends on both the magnitude of the noise “forces” *and* the time averaged temperature anisotropy induced by the lattice.