Ion Beam Optics

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Previous lectures

1. Principle of least action
   - Canonical equations
   - Canonical transformations
   - Reversibility
   - Liouville's theorem in discrete (6*N*-dim.) and in continuous (6-dim.) description
   - Generalized canonical transformations (i.e., transformations that include mappings of time)

Previous lectures (cont.)

2. Linear beam optics without space charge
   - Single particle transformations
     - Quadrupole
     - Drift
     - Bending magnet
   - Hill's equation
   - "hard edge" model
   - Ellipse transformation formalism

Previous lectures (cont.)

3. Theory of alternating gradient focusing
   - Eigenellipse parameters
   - Stability considerations
   - Phase advance
   - Matching
4. Hill's equation
   - General solution
   - Envelope equation
5. Statistical description
   - Definition of "moments"
   - RMS envelope equation
   - RMS emittance
   - Condition for conservation of RMS emittance

Actual lectures

6. Beam optics with space charge
   – Single particle equations with space charge
   – Moment equations
   – Stability analysis of the K-V equations
   – Beam matching under space charge conditions
7. Emittance growth effects
   – Emittance and field energy relations
   – Phase space probability densities for unbunched beams
   – Bunched beams, equipartitioning
8. Stationary phase space distributions
   – General theory
   – self-consistent definition of KV, “Waterbag”, and Gaussian distributions
   – “Stationary” beams under periodic focusing

Actual lectures (cont.)

9. Charge “granularity” effects
   – Fokker-Planck model
   – Time reversal effects
   – Definition of entropy
   – Stationary phase space distributions
10. Moment analysis of the Fokker-Planck equation
    – Generalized envelope equation
    – Beam temperature
    – Reversible and irreversible emittance growth
    – Intra-beam scattering
11. “Fokker-Planck effects” in computer simulations

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6. Beam Optics with space charge

6.1. Single particle equations of motion

So far, we have only treated *single particle dynamics*, i.e., the motion of each particle is *not* influenced by other beam particles.

➔This is an approximation as charged particles always produce a self field!

What changes if we take into account the self fields?

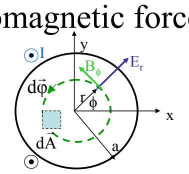
- The description of the system's dynamics is now more complicated:
  - Problem of "self-consistency" arises: the particle positions depend on the total force, but at the same time, the total force depends on the particle positions.
- The self-field forces induce additional resonance effects.
- We get additional sources of emittance growth.
- We must distinguish fine-grained description where all *individual* particles are taken into account from a smoothed description where the self fields are *continuous* functions of the spatial coordinates.

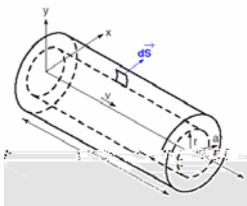
6. Beam optics with space charge
6.1. Single particle equations of motion

Simple model of an unbunched beam: homogeneously filled moving cylinder of radius a.

We have two components of electromagnetic forces:

- Repulsive electric force F_e
- Attractive magnetic force F_m





Smooth description:

$$\iiint \operatorname{div} \vec{E} dV = \iint \vec{E} d\vec{S}, \qquad \iint \operatorname{curl} \vec{B} d\vec{A} = \oint \vec{B} d\vec{s}$$

to the beam model of *constant density* both longitudinally and transversely:

$$\operatorname{div} \vec{E} = \frac{\rho}{\varepsilon_0}, \quad \operatorname{curl} \vec{B} = \mu_0 \vec{j}, \quad \rho = \frac{I}{\pi a^2 \beta c}, \quad \vec{j} = \rho \beta c \, \vec{e}_z$$

6. Beam optics with space charge
6.1. Single particle equations of motion

For the simplified electric and magnetic fields

$$\vec{E} = E_r \, \vec{e}_r, \quad \vec{B} = B_\varphi \, \vec{e}_\varphi$$

we find in cylindrical coordinates the forms

$$dV = r dr d\varphi dz, \quad \vec{E} d\vec{S} = E_r \, r d\varphi dz, \quad \vec{E} \perp \vec{S}$$

$$\vec{B} d\vec{s} = B_\varphi \, r d\varphi, \quad \vec{j} d\vec{A} = \rho c \beta \, r dr d\varphi, \quad \vec{s} \parallel \vec{B}, \; \vec{j} \perp \vec{A}$$

and hence the field components for a beam of radius a $\quad (\mu_0 = 1/\varepsilon_0 c^2)$

$$E_r = \frac{\rho}{2\varepsilon_0} r = \frac{I}{2\pi\varepsilon_0 \beta c} \frac{r}{a^2}, \quad B_\varphi = \tfrac{1}{2} \mu_0 \rho \beta c \, r = \frac{I}{2\pi\varepsilon_0 c^2} \frac{r}{a^2}$$

6. Beam optics with space charge
6.1. Single particle equations of motion

The Lorentz force equation

$$\vec{F} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

takes on the particular form

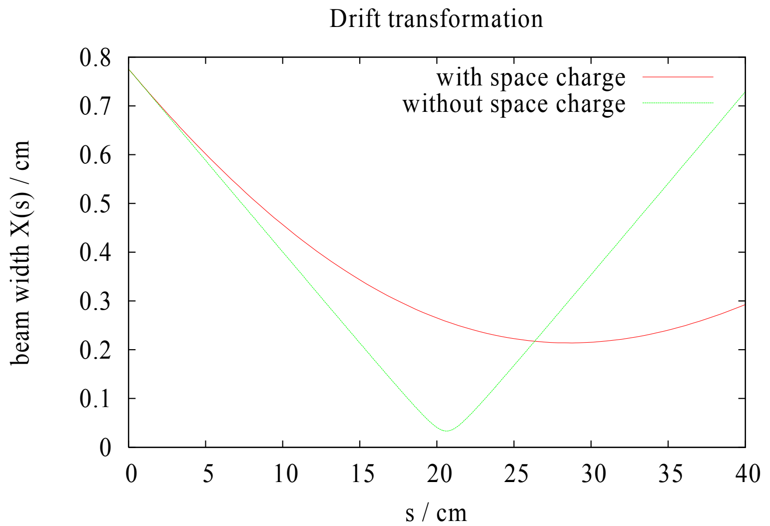
$$F_r(r) = e\left(E_r - \beta c\, B_\varphi\right) = \frac{eI}{2\pi\varepsilon_0 \beta c}\left(1-\beta^2\right)\frac{r}{a^2}$$
$$= \frac{eI}{2\pi\varepsilon_0 \beta c \gamma^2}\,\frac{r}{a^2}$$

The magnetic field yields the factor $1/\gamma^2$

Generalization to uniform *elliptic* beams of half axes X, Y:

$$F_x = \frac{eI}{\pi\varepsilon_0 c\beta\gamma^2}\,\frac{x}{X\,(X+Y)}, \qquad F_y = \frac{eI}{\pi\varepsilon_0 c\beta\gamma^2}\,\frac{y}{Y\,(X+Y)}$$

6. Beam optics with space charge
6.1. Single particle equations of motion



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Single particle equations of motion that include space-charge forces:

$$x'' + k_{x,\mathrm{ext}}^2(s)\,x - \frac{1}{m\gamma c^2\beta^2}F_x(x,y,s) = 0$$

$$y'' + k_{y,\mathrm{ext}}^2(s)\,y - \frac{1}{m\gamma c^2\beta^2}F_y(x,y,s) = 0$$

We insert the linear space-charge forces of the simplified model:

$$x'' + \left(k_{x,\mathrm{ext}}^2(s) - \frac{2K}{X(X+Y)}\right)x = 0, \qquad K = \frac{2eI}{4\pi\varepsilon_0 mc^3\beta^3\gamma^3}$$

$$y'' + \left(k_{y,\mathrm{ext}}^2(s) - \frac{2K}{Y(X+Y)}\right)y = 0. \qquad K : \text{"generalized perveance"}$$

K is a dimensionless measure for the strength of the space charge. *K* can be regarded as the "scaled current".

6. Beam optics with space charge
6.1. Single particle equations of motion

Linear model: oscillation frequency is the same for all particles

$$k_{x,\mathrm{ext}}^2 = k_{y,\mathrm{ext}}^2 = k_0^2 = \text{const.} \quad \Rightarrow \quad \sigma_0 = k_0 S \quad : \text{zero current phase advance}$$

$$k^2 \stackrel{\text{Def}}{=} k_0^2 - \frac{K}{a^2} \quad \Rightarrow \quad \sigma = kS \quad : \text{depressed tune}$$

The reduction of the phase advance $\sigma_0 \rightarrow \sigma$ due to space charge forces is often referred to as the "Laslett tune shift".

The ratio σ/σ_0 is a good measure for the relative strength of the space charge compared to the external focusing strength.

For synchrotrons, we generally use the terms "tune" $Q_{0,x,y} = \sigma_{0,x,y}/2\pi$ i.e., the number of oscillations per turn and the corresponding "incoherent tune shift" $\Delta Q_{x,y}$. We will now derive a formula to estimate $\Delta Q_{x,y}$ in a synchrotron in terms of the beam parameters.

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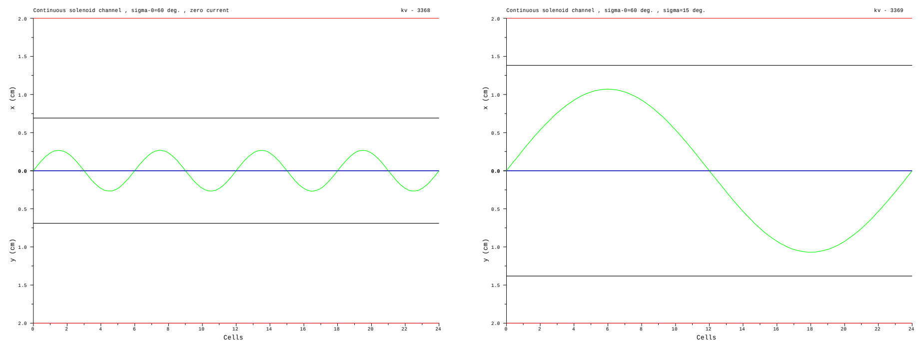
6. Beam optics with space charge
6.1. Single particle equations of motion



Particle oscillation frequency for zero current of $\sigma_0 = 60^{\circ}$ (left), depressed to $\sigma = 15^{\circ}$ due to space charge (right).

6. Beam optics with space charge
6.1. Single particle equations of motion

$$k^2 = k_0^2 - \frac{K}{a^2} = (k_0 - \Delta k)^2$$

$$\approx k_0^2 - 2k_0\,\Delta k, \quad \Delta k \ll k_0,\, \Delta Q_{x,y} \ll Q_{0,x,y}$$

$$\frac{\Delta k}{k_0} = \frac{\Delta Q_{x,y}}{Q_{0,x,y}} \approx -\frac{1}{2}\frac{K}{k_0^2 a^2} \quad : \quad \text{relative tune shift}$$

For a "continuous focusing system", we know from Sec. 3

$$a^2 = \varepsilon_x \beta_x, \quad \beta_x = \frac{S}{\sigma_x} = \frac{R}{Q_x}$$

With R the radius of the synchrotron, this yields

$$\Delta Q_{x,y} \approx -\frac{1}{2} K \frac{R}{\varepsilon_{x,y}}, \quad I = \frac{Nec\beta}{2\pi R}$$

or, in terms of the number of particles N along the ring circumference:

$$\Delta Q_{x,y} \approx -\frac{r_0}{2\pi\beta^2\gamma^3}\frac{N}{\varepsilon_{x,y}}, \quad r_0 = \frac{e^2}{4\pi\varepsilon_0 m c^2}$$

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General, nonlinear case (simulation codes):

$$x'' + k_{x,\mathrm{ext}}^2(s)\,x - \frac{e}{mc^2\beta^2\gamma^3}E_x(x,y,s) = 0$$

$$y'' + k_{y,\mathrm{ext}}^2(s)\,y - \frac{e}{mc^2\beta^2\gamma^3}E_y(x,y,s) = 0$$

with the space-charge fields as the solution of Poisson's equation

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \frac{1}{\varepsilon_0}\rho(x,y,s)$$

$\rho(x,y,s)$ may be either the fine-grained field of all particles or the smooth field due to a phase space probability density

General problem for *analytic* treatment: the functional form of the charge density ρ is usually *not* conserved and hence unknown!

➔ problem of stationary phase space distributions (treated later).

6. Beam Optics with space charge

6.2. Moment equations

Usually, we not particularly interested in *single particles.*

➔"moment equations"

If the complete information on N particles is given, the first moment and the second central moment in x are calculated via:

$$\overline{x}(s) = \frac{1}{N}\sum_{i=1}^{N} x_i(s), \quad \overline{x^2}(s) = \frac{1}{N}\sum_{i=1}^{N}\left(x_i(s) - \overline{x}(s)\right)^2$$

If we succeed in deriving closed equations of motion for the second moments, the system's description is considerably simplified.

The equations of motion for the moments are obtained by calculating their time derivatives and inserting the single particle equations of motion.

In the particle description, the other second moments are defined as

$$\overline{x^2}(s) = \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \overline{x}\right)^2 = \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \left(\overline{x}\right)^2$$

$$\overline{xx'}(s) = \frac{1}{N}\sum_{i=1}^{N}\left(x_i - \overline{x}\right)\left(x_i' - \overline{x'}\right) = \frac{1}{N}\sum_{i=1}^{N} x_i x_i' - \overline{x}\cdot\overline{x'}$$

$$\overline{x'^2}(s) = \frac{1}{N}\sum_{i=1}^{N}\left(x_i' - \overline{x'}\right)^2 = \frac{1}{N}\sum_{i=1}^{n} x_i'^2 - \left(\overline{x'}\right)^2$$

The equation of motion of the first moment is simply the single particle equation for the center-of-mass motion.

We are interested in the equations of motion of the second moments:

$$\frac{d}{ds}\overline{x^2}(s) = \frac{2}{N}\sum_{i=1}^{N}\left(x_i - \overline{x}\right)\left(x_i' - \overline{x'}\right) = 2\,\overline{xx'}(s) \qquad \text{(M1)}$$

Same procedure for the derivatives of the other second moments:

$$\frac{d}{ds}\overline{xx'}(s) = \overline{x'^2}(s) - k^2_{x,\text{ext}}(s)\,\overline{x^2}(s) + \frac{e}{mc^2\beta^2\gamma^3}\overline{xE_x}(s) \qquad \text{(M2)}$$

$$\frac{d}{ds}\overline{x'^2}(s) = -2k^2_{x,\text{ext}}(s)\,\overline{xx'}(s) + \frac{2e}{mc^2\beta^2\gamma^3}\overline{x'E_x}(s) \qquad \text{(M3)}$$

Here, we have replaced x'' according to the single particle equation of motion.

Note that the moments that are related to the electric field E_x involve higher than second moments in general. In that case, the moment equations lead to an infinite hierarchy. For the linear model, only second moments appear ➔ closed system of equations.

The "smooth description" is based on a phase space probability density $f(x,x',y,y',s)$, i.e., $f\,d\tau$ is the *probability* for finding a particle inside a small volume $d\tau$ around (x,x',y,y') at s.

This is the "smooth version" of the Liouville theorem

$$\frac{df}{ds} = 0 \quad \Leftrightarrow \quad \frac{\partial f}{\partial s} + x'\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + x''\frac{\partial f}{\partial x'} + y''\frac{\partial f}{\partial y'} = 0$$

Inserting the "smooth" single particle equation of motion

$$x'' + k^2_{x,\text{ext}}(s)\,x - \frac{e}{mc^2\beta^2\gamma^3}E_x^{\text{smooth}}(x,y,s) = 0$$

we obtain the Vlasov equation, i.e. the closed equation of motion for f

$$\frac{\partial f}{\partial s} + x'\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} - \left(k^2_{x,\text{ext}}(s)\,x - \frac{e}{mc^2\beta^2\gamma^3}E_x^{\text{smooth}}(x,y,s)\right)\frac{\partial f}{\partial x'}$$
$$- \left(k^2_{y,\text{ext}}(s)\,y - \frac{e}{mc^2\beta^2\gamma^3}E_y^{\text{smooth}}(x,y,s)\right)\frac{\partial f}{\partial y'} = 0, \quad \text{div}\vec{E} = \frac{1}{\varepsilon_0}\iint f\,dx'dy'$$

Usually, we not particularly interested in the detailed information contained in $f(x,x',y,y',s)$ ➔ moment equations.

The first moment and the second central moment in x are given by:

$$\overline{x}(s) = \int x f(x,x',y,y',s)\,d\tau, \qquad \overline{x^2}(s) = \int (x-\overline{x})^2 f(x,x',y,y',s)\,d\tau$$

To derive the equations of motion of the moments, we evaluate the s-derivatives similar to:

$$\frac{d}{ds}\overline{x^2}(s) = \int (x-\overline{x})^2 \frac{\partial}{\partial s} f(x,x',y,y',s)\,d\tau$$

The moments are then calculated inserting the Vlasov equation.

The resulting set of equations agrees with single particle approach.

➔ Second moment analysis of the Vlasov equation.

Applications of the moment equations

We insert equation (M2) into equation (M1):

$$\tilde{x}(s) \overset{\text{Def}}{=} \sqrt{\overline{x^2}(s)}, \qquad \tilde{\varepsilon}_x^2(s) \overset{\text{Def}}{=} \overline{x^2}\,\overline{x'^2} - \left(\overline{xx'}\right)^2 \equiv \frac{1}{2N^2}\sum_{i,j=1}^{N}\left(x_i x_j' - x_j x_i'\right)^2$$

"RMS emittance"

$$\frac{d^2}{ds^2}\tilde{x}(s) + k_{x,\text{ext}}^2(s)\,\tilde{x} - \frac{e}{mc^2\beta^2\gamma^3}\frac{\overline{xE_x}(s)}{\tilde{x}} - \frac{\tilde{\varepsilon}_x^2(s)}{\tilde{x}^3} = 0$$

In this form, the last equation does not seem to be of much help since it contains two unknown functions of s. Let us study them now.

For a beam with elliptical symmetry *in real space*, we can show that

$$\overline{xE_x}(s) = \frac{I}{4\pi\varepsilon_0 c\beta}\frac{\tilde{x}}{\tilde{x}+\tilde{y}} \quad\Leftrightarrow\quad \rho(x,y) = \rho\left(\frac{x^2}{X^2} + \frac{y^2}{Y^2}\right)$$

$$\frac{d^2}{ds^2}\tilde{x}(s)+k_{x,\mathrm{ext}}^2(s)\,\tilde{x}-\frac{\frac{1}{2}K}{\tilde{x}+\tilde{y}}-\frac{\tilde{\varepsilon}_x^2(s)}{\tilde{x}^3}=0$$

As expected, the space charge couples the *x*- and the *y*-planes.

The only remaining unknown function is the RMS emittance. In order to study its time behavior, we calculate its derivative:

For a 2-dimensional *uniform* density $\rho(x,y)$, we have

$$\tilde{x}^2 = \frac{\iint x^2 \rho(x,y)dxdy}{\iint \rho(x,y)dxdy}, \qquad \rho(x,y)=1, \quad x = r\cos\varphi, \quad dxdy = rdrd\varphi$$

$$= \frac{\int_{r=0}^{R}\int_{\varphi=0}^{2\pi} r^2\cos^2\varphi\, rdrd\varphi}{\int_{r=0}^{R}\int_{\varphi=0}^{2\pi} rdrd\varphi} = \frac{\pi\int_{r=0}^{R} r^3 dr}{2\pi\int_{r=0}^{R} r\,dr} = \frac{R^2}{4}$$

and hence for an elliptic beam of uniform density transversely:

$$\tilde{x} = \tfrac{1}{2} X$$

Problem: does an elliptic beam of uniform density stay elliptic and uniform? We will answer this question later.

We will later treat second and higher order moments for other phase space density functions.

In terms of $X(s)$, the RMS K-V envelope equations are simply referred to as the **K-V envelope equations**

$$\frac{d^2}{ds^2}X(s) + k_{x,\text{ext}}^2(s)\,X - \frac{2K}{X+Y} - \frac{\varepsilon_x^2}{X^3} = 0$$

$$\frac{d^2}{ds^2}Y(s) + k_{y,\text{ext}}^2(s)\,Y - \frac{2K}{X+Y} - \frac{\varepsilon_y^2}{Y^3} = 0$$

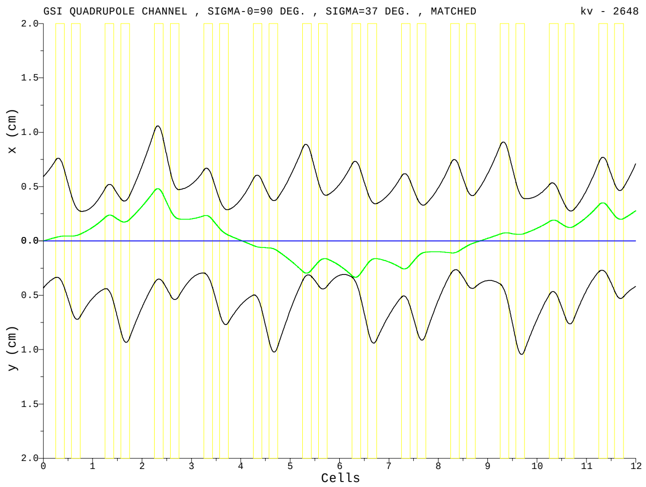
with the total emittances defined as

$$\varepsilon_x = 4\tilde{\varepsilon}_x, \qquad \varepsilon_y = 4\tilde{\varepsilon}_y$$

For uniform density, the total emittance is the “area” of the 2-dim. x,x′- and y,y′-phase-space projections.

The K-V equations are an important tool for the first-order design of high-current lattices. Let us study some examples.

6. Beam optics with space charge
6.2. Moment equations



Mismatched beam in a periodic quadrupole channel

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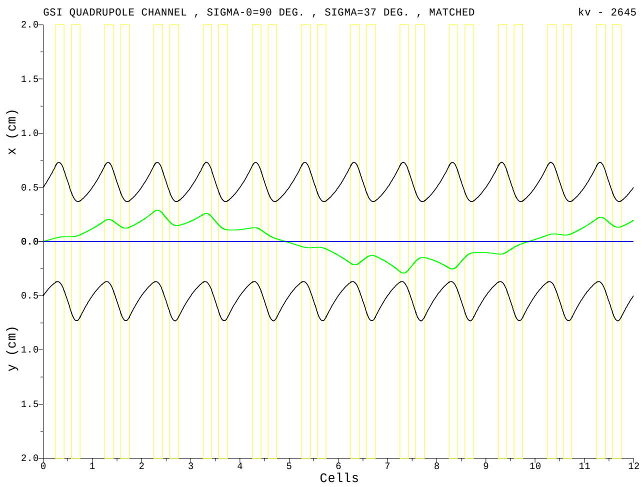
6. Beam optics with space charge
6.2. Moment equations



Matched beam in a periodic quadrupole channel

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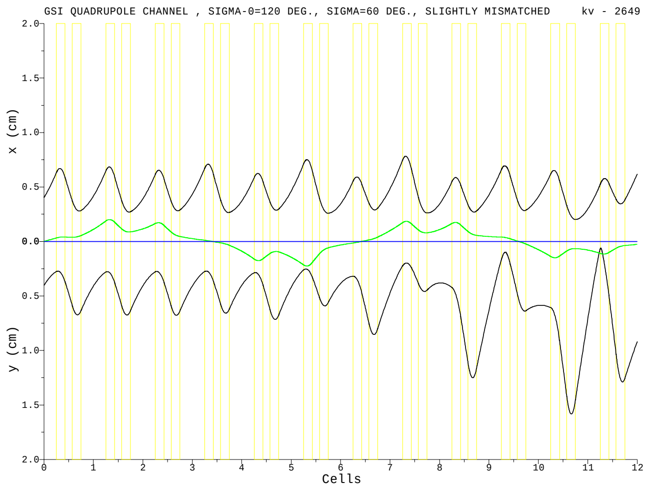
6. Beam optics with space charge
6.2. Moment equations



Unstable (chaotic) beam in a periodic quadrupole channel

6. Beam optics with space charge
6.2. Moment equations

- The following questions arise:
  - Under which conditions is the beam transformation stable ?
  - How do we obtain the matched solution ? (remember: the Courant-Synder theory of alternating-gradient synchrotrons of Sec. 3.1 applies for zero current only!)
- We will see that both questions are closely related.
- Thus, the next topic will be the stability analysis of the K-V envelope equations.

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6. Beam Optics with space charge

6.3. Stability analysis of the K-V equations

We first note that the system of K-V equations can be regarded as the *canonical equations* of the Hamiltonian $H_{K\text{-}V}$

$$H_{\text{K-V}}\left(X,Y,P_x,P_y,s\right)=\tfrac{1}{2}\left(P_x^2+P_y^2\right)+\tfrac{1}{2}\left(k_x^2(s)\,X^2+\frac{\varepsilon_x^2}{X^2}+k_y^2(s)\,Y^2+\frac{\varepsilon_y^2}{Y^2}-4K\ln(X+Y)\right)$$

➔ The solutions of the K-V equations have the hamiltonian properties: symplectic geometry, conservation of volume form ("Liouville"). We will make use of these properties later in our stability analysis.

Parametric resonance: if a system parameter depends on *s* (or *t*), the system can be *unstable*, even if it is stable for each fixed value of the parameter.

For the stability analysis, we compare the behavior of *neighboring solutions*:

$$X(s) = X_0(s) + x(s), \quad Y(s) = Y_0(s) + y(s), \quad x(0) \ll X_0(0),\ y(0) \ll Y_0(0)$$

wherein $X_0(s)$ and $Y_0(s)$ denote the reference envelopes, and $x(s)$, $y(s)$ the — at least initially — small deviations.

We insert $X(s)$ and $Y(s)$ into the K-V equations and apply the approximations

$$\left(1 + \frac{x+y}{X_0 + Y_0}\right)^{-1} \approx 1 - \frac{x+y}{X_0+Y_0}, \quad \left(1 + \frac{x}{X_0}\right)^{-3} \approx 1 - 3\frac{x}{X_0}$$

This yields a *linear* coupled set of equations for the perturbations $x(s)$ and $y(s)$

6. Beam optics with space charge
6.3. Stability analysis of the K-V equations

$$x''(s) + \left(k_x^2(s) + \frac{2K}{(X_0+Y_0)^2} + \frac{3\varepsilon_x^2}{X_0^4}\right)x(s) + \frac{2K}{(X_0+Y_0)^2}\,y(s) = 0$$

$$y''(s) + \left(k_y^2(s) + \frac{2K}{(X_0+Y_0)^2} + \frac{3\varepsilon_y^2}{Y_0^4}\right)y(s) + \frac{2K}{(X_0+Y_0)^2}\,x(s) = 0$$

Again, the coupling is induced by a non-vanishing K. We can rewrite these two second-order equations as a linear system of four first-order equations:

$$\frac{d}{ds}\begin{pmatrix} z_1(s) \\ z_2(s) \\ z_3(s) \\ z_4(s) \end{pmatrix} = A(s)\begin{pmatrix} z_1(s) \\ z_2(s) \\ z_3(s) \\ z_4(s) \end{pmatrix}, \qquad A(s) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_1(s) & -a_0(s) & 0 & 0 \\ -a_0(s) & -a_2(s) & 0 & 0 \end{pmatrix}$$

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or in vector form $\vec{z}\,'(s) = A(s)\,\vec{z}(s)$

with
$$z_1 \equiv x, \quad z_2 \equiv y, \quad z_3 \equiv x', \quad z_4 \equiv y',$$
$$a_0(s) = \frac{2K}{[X_0(s) + Y_0(s)]^2}$$
$$a_1(s) = k_x^2(s) + \frac{3\varepsilon_x^2}{X_0^4(s)} + a_0(s),$$
$$a_2(s) = k_y^2(s) + \frac{3\varepsilon_y^2}{Y_0^4(s)} + a_0(s),$$

➔ If the focusing functions $k_{x,y}(s)$ are periodic with period S, and the envelopes $X_0(s)$ and $Y_0(s)$ have the same periodicity (matched solutions), then the coefficients $a_i(s)$ and hence $A(s)$ are also S-periodic.

Rough simplification: external forces on the beam are symmetric and not s-dependent
$$k_x(s) = k_y(s) = k_0 = \text{const.}$$
A *matched* beam has then *constant* envelopes:
$$X_0(s), Y_0(s) = \text{const.}$$
If, furthermore, the emittances agree, then
$$\varepsilon_x = \varepsilon_y = \varepsilon \quad \Rightarrow \quad X_0 = Y_0 = a$$
Since $X_0'' = Y_0'' = 0$, the K-V envelope equations simplify to:
$$k_0^2 a - \frac{K}{a} - \frac{\varepsilon^2}{a^3} = 0$$

For $a=a_{max}$, this yields Reiser's formula for the maximum beam current $K=K_{max}$

With the previously defined depressed tune k

$$k^2 \stackrel{\mathrm{Def}}{=} k_0^2 - \frac{K}{a^2},$$

the coupled set of equations for the envelope perturbations is given by

$$x''(s) + \tfrac{1}{2}\left(3k_0^2 + 5k^2\right)x(s) + \tfrac{1}{2}\left(k_0^2 - k^2\right)y(s) = 0$$
$$y''(s) + \tfrac{1}{2}\left(3k_0^2 + 5k^2\right)y(s) + \tfrac{1}{2}\left(k_0^2 - k^2\right)x(s) = 0$$

We can decouple these equations by introducing the new functions

$$z_{\mathrm{odd}}(s) \stackrel{\mathrm{Def}}{=} x(s) - y(s), \qquad z_{\mathrm{even}}(s) \stackrel{\mathrm{Def}}{=} x(s) + y(s)$$

→ $z_{\mathrm{odd}}(s)$ describes the 180°-out-of-phase ("odd") mode of the perturbation oscillation, $z_{\mathrm{even}}(s)$ describes the in-phase ("even") mode.

The equations for $z_{\mathrm{odd}}(s)$ and $z_{\mathrm{even}}(s)$ are

$$z''_{\mathrm{odd}}(s) + k_{\mathrm{odd}}^2 z_{\mathrm{odd}}(s) = 0, \qquad k_{\mathrm{odd}}^2 = k_0^2 + 3k^2$$
$$z''_{\mathrm{even}}(s) + k_{\mathrm{even}}^2 z_{\mathrm{even}}(s) = 0, \qquad k_{\mathrm{even}}^2 = 2k_0^2 + 2k^2$$

Depending on both the zero current tune and the tune depression, the frequency of the two fundamental modes are *different!*

→ A general mismatch oscillation is a *beat wave* of both modes.

The wave numbers $k_{\mathrm{odd,even}}$ are equivalent to the phase advances

$$\phi_{\mathrm{odd}} = k_{\mathrm{odd}} S = \sqrt{\sigma_0^2 + 3\sigma^2}, \qquad \phi_{\mathrm{even}} = k_{\mathrm{even}} S = \sqrt{2\sigma_0^2 + 2\sigma^2}$$

which have the limit values

$$\lim_{I\to 0} \phi_{\mathrm{odd}} = \lim_{I\to 0} \phi_{\mathrm{even}} = 2\sigma_0, \qquad \lim_{I\to\infty} \phi_{\mathrm{odd}} = \sigma_0, \; \lim_{I\to\infty} \phi_{\mathrm{even}} = \sqrt{2}\,\sigma_0$$

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Numerical example: continuous focusing channel with

$$\sigma_0 = 60°, \quad \sigma = 15°$$

$$\Rightarrow \quad \phi_{\text{odd}} = \sqrt{\sigma_0^2 + 3\sigma^2} = 15\sqrt{19}° \cong 65.4°$$

$$\phi_{\text{even}} = \sqrt{2\sigma_0^2 + 2\sigma^2} = 15\sqrt{34}° \cong 87.5°$$

For one mismatch oscillation of the odd mode, we need, therefore

$$N_{\text{odd}} = 360° / 65.4° = 5.5 \text{ cells}$$

Similarly, for one mismatch oscillation of the even mode, we need

$$N_{\text{even}} = 360° / 87.5° = 4.1 \text{ cells}$$

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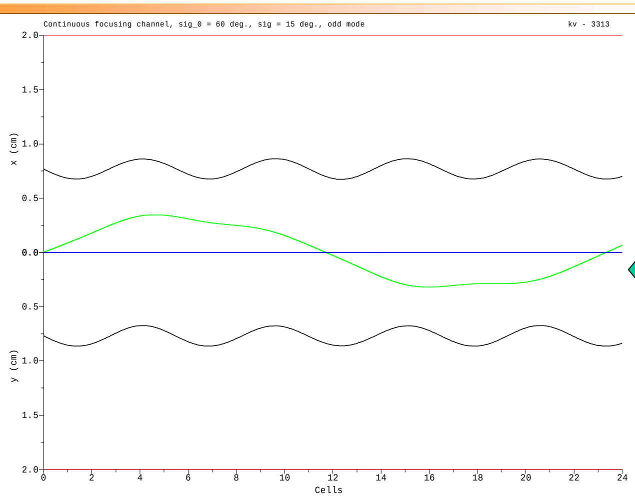

The green line shows the oscillation of a single particle in the x,s plane.

Odd mismatch mode: 5.5 cells per mismatch oscillation at $\sigma_0 = 60°$, $\sigma = 15°$

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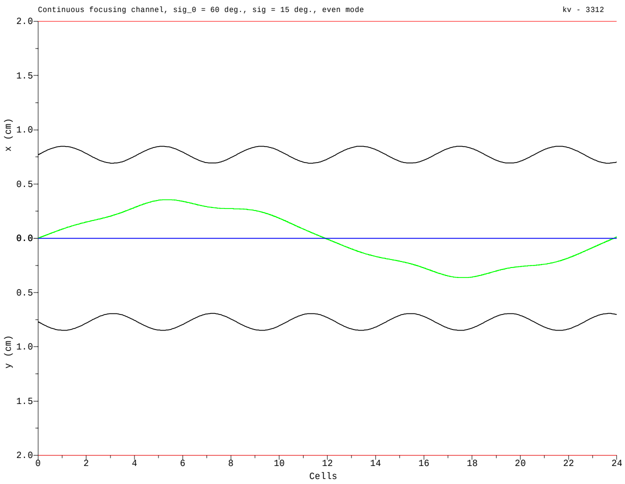

Even mismatch mode: 4.1 cells per mismatch oscillation at $\sigma_0 = 60°$, $\sigma = 15°$

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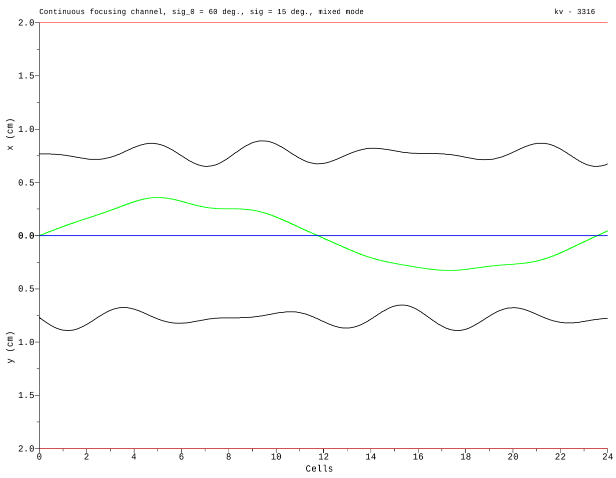

Mixed mismatch mode at $\sigma_0 = 60°$, $\sigma = 15°$: beat wave of fundamental modes

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General case: let $Z(s)$ be the matrix of *linear independent solutions* of the *periodic* system

$$\vec{z}\,'(s) = A(s)\ \vec{z}(s)$$

with $Z(0)=E$. This means:

$$Z'(s) = A(s)\,Z(s), \quad A(s+S) = A(s), \quad Z(0) = E$$

Then the matrix $Z(S)$ is called the *monodromy matrix* or the *mapping at a period*. It maps every state at $s=0$ into the corresponding state at $s=S$:

$$\vec{z}(S) = Z(S)\,\vec{z}(0)$$

Theorem: the solution matrix at $s+nS$ has the representation:

$$Z(s+nS) = Z(s)\ Z^n(S), \quad n \in \mathbb{N}$$

Proof: if $Z(s)$ is a solution matrix, then $Z(s+nS)$ is also a solution matrix:

$$s \mapsto s+nS: \quad Z'(s+nS) = A(s+nS)\,Z(s+nS)$$
$$= A(s)\,Z(s+nS)$$

For $Z(s)$ a solution matrix and C a regular *constant* matrix, we get:

$$[Z(s)C]' = Z'(s)C = [A(s)Z(s)]C = A(s)[Z(s)C]$$

This means that $Z(s)C$ is also a solution matrix

Set

$$C = Z^n(S) \quad \Rightarrow \quad Z(s)Z^n(S) \text{ is a solution matrix}$$
$$\Rightarrow \quad Z(s+nS) = Z(s)\,Z^n(S) \text{ by induction}$$

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Theorem: for every eigenvalue λ of $Z(S)$ there exists a solution $\vec{\varsigma}(s)$ with

$$\vec{\varsigma}(s+nS) = \lambda^n \, \vec{\varsigma}(s)$$

Proof: With $\vec{\varsigma}(0)$ denoting the eigenvector for λ, we have by definition:

$$Z(S)\vec{\varsigma}(0) = \lambda\vec{\varsigma}(0)$$

and therefore

$$\begin{aligned}\vec{\varsigma}(s+nS) &= Z(s+nS)\vec{\varsigma}(0)\\ &= Z(s)\,Z^n(S)\vec{\varsigma}(0)\\ &= Z(s)\,Z^{n-1}(S)\,Z(S)\vec{\varsigma}(0)\\ &= Z(s)\,Z^{n-1}(S)\,\lambda\,\vec{\varsigma}(0)\\ &\vdots\\ &= Z(s)\,\lambda^n\,\vec{\varsigma}(0) = \lambda^n\,Z(s)\vec{\varsigma}(0)\\ &= \lambda^n\,\vec{\varsigma}(s)\end{aligned}$$

6. Beam optics with space charge
6.3. Stability analysis of the K-V equations

The fact that our system is *hamiltonian* allows an important simplification of our stability analysis.

To this end, we first define the $2n$fi$2n$ *symplectic unit matrix I*:

$$I = \begin{pmatrix} O & -E \\ E & O \end{pmatrix},$$

with E denoting the $n$fi$n$ *unit matrix,* and O the $n$fi$n$ *zero matrix.*

In the particular case of our *4*fi*4* system, the symplectic unit matrix is:

$$I = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The *symplectic unit matrix I* has the following properties:

$$I^T = -I, \quad I^{-1} = I^T, \quad I^2 = -E$$

Lemma 1: The matrix $I\,A(s)$ is symmetric.

Proof:

$$I\,A(s) = \begin{pmatrix} a_1(s) & a_0(s) & 0 & 0 \\ a_0(s) & a_2(s) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lemma 2: For two linear independent solutions of $z'=A(s)\,z$, the following scalar product is a constant of motion:

$$\left(\vec{z}_1, I\,\vec{z}_2\right) = \text{const.}$$

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Proof:

$$\begin{aligned}
\frac{d}{ds}\left(\vec{z}_1, I\,\vec{z}_2\right) &= \left(\vec{z}_1', I\,\vec{z}_2\right) + \left(\vec{z}_1, I\,\vec{z}_2'\right) \\
&= \left(A\vec{z}_1, I\,\vec{z}_2\right) + \left(\vec{z}_1, IA\,\vec{z}_2\right) \\
&= \left(\vec{z}_1, A^T I\,\vec{z}_2\right) + \left(\vec{z}_1, IA\,\vec{z}_2\right) \\
&= \left(\vec{z}_1, \left(A^T I + IA\right)\vec{z}_2\right) \\
&= \left(\vec{z}_1, \left(IA - A^T I^T\right)\vec{z}_2\right) \\
&= \left(\vec{z}_1, \left(IA - (IA)^T\right)\vec{z}_2\right) = 0
\end{aligned}$$

as IA is symmetric, i.e. equals its transpose. We are now prepared to proof the main theorem.

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We finally conclude:

$$p(\lambda_1)=0 \text{ for } \lambda_1 \neq 0 \quad \Rightarrow \quad p(1/\lambda_1)=0$$

As an immediate consequence of the fact that the characteristic polynomial is *real*, we find:

Theorem: If λ_1 is a complex eigenvalue, then the conjugate complex value $\lambda_1{}^*$ is also an eigenvalue.

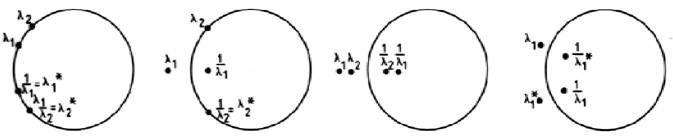
$$\lambda_1 \in \mathbb{C} \text{ is eigenvalue of } Z(S) \quad \Leftrightarrow \quad \lambda_1^* \in \mathbb{C} \text{ is eigenvalue of } Z(S)$$

Proof:

$$p(\lambda)=(\lambda-\lambda_1)\ldots(\lambda-\lambda_n), \quad \lambda_1,\ldots,\lambda_n \in \mathbb{C}$$

$$\lambda_i \in \mathbb{C} \quad \Rightarrow \quad (\lambda-\lambda_i)(\lambda-\lambda_i^*) \in \mathbb{R} \quad \Leftrightarrow \quad p(\lambda) \in \mathbb{R}$$

In summary, we have only the following *four* possibilities for the location of the eigenvalues in the complex plane:



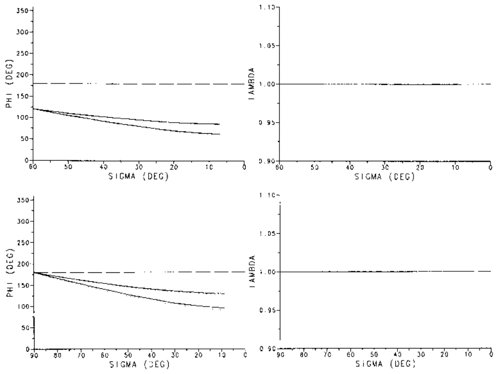
Recall: The matrix $Z(S)$ (the "monodromy matrix") maps an initial perturbation through one cell. With ζ an eigenvector of $Z(S)$ and λ the corresponding eigenvalue, we have $\vec{\varsigma}(nS)=\lambda^n\varsigma(0)$. Consequently

$$|\lambda| \overset{!}{=} 1$$

If an eigenvalue with $|\lambda| \neq 1$ exists, then a growing mode with rate $\lambda > 1$ exists → exponential growth as long $x,y \subset X_0,Y_0$!

is a necessary condition for the stability of a *periodic system* against mismatch perturbations (which can never be avoided in real systems).

Example 1: we plot $\phi_{\text{odd,even}}$ and $|\lambda|$ for *fixed* $\sigma_0 = 60°$ and $\sigma_0 = 90°$ against diminishing σ, i.e., against increasing beam current I:



All eigenvalues lie on the unit circle ($|\lambda|=1$).

All eigenvalues lie on the unit circle ($|\lambda|=1$).

The dotted lines show the corresponding values for a *continuous* channel

We observe: with $\sigma_0 \leq 90°$, the system is *always stable* since $|\lambda| = 1$.

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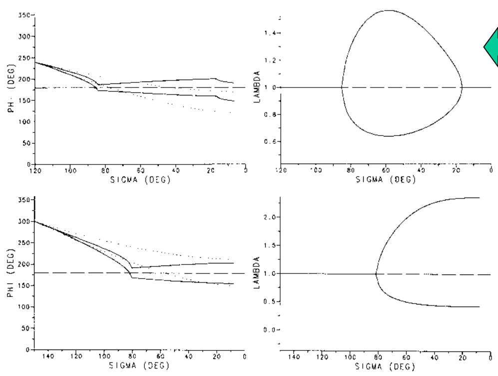
Example 2: we plot $\phi_{\text{odd,even}}$ and $|\lambda|$ for *fixed* $\sigma_0 = 120°$ and $\sigma_0 = 150°$ against diminishing σ, i.e., against increasing beam current I:



In a *continuous* channel (dotted lines) no instabilities occur

For $\sigma_0 = 120°$, $\sigma = 60°$, we get a growth of the perturbations $x(s)$,$y(s)$ of ≈ 1.56 per cell !

We observe: with $\sigma_0 > 90°$, the system gets *unstable* for $\sigma < 90°$.

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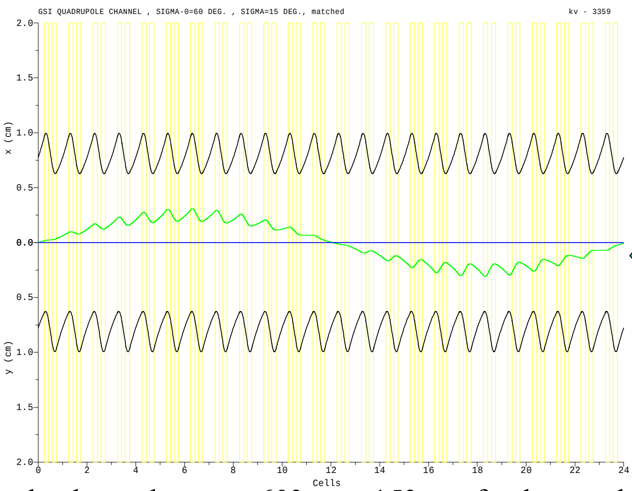

The green line shows the oscillation of a single particle in the x,s plane.

Quadrupole channel at $\sigma_0 = 60°$, $\sigma = 15°$: perfectly matched beam

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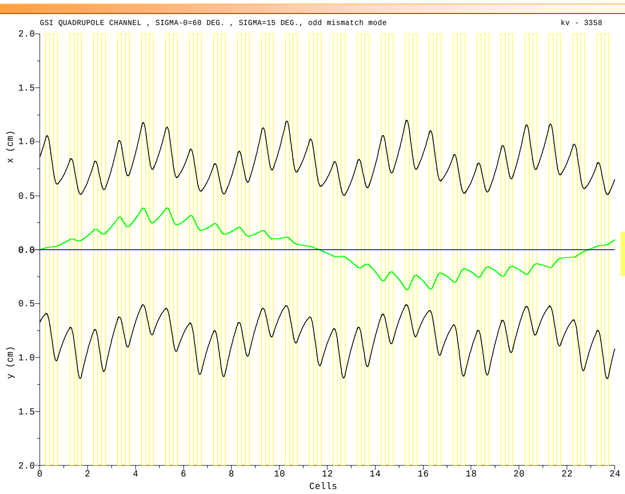

$N_{\mathrm{odd}} = 5.5$ cells

Quadrupole channel at $\sigma_0 = 60°$, $\sigma = 15°$: odd mismatch mode

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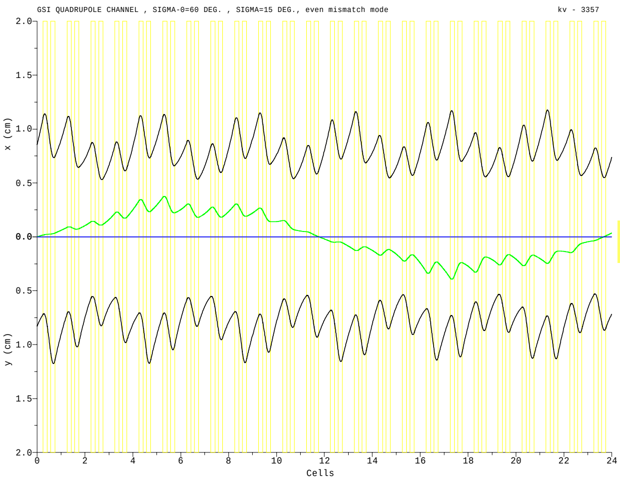

$N_{\text{even}} = 4.1$ cells

Quadrupole channel at $\sigma_0 = 60°$, $\sigma = 15°$: even mismatch mode

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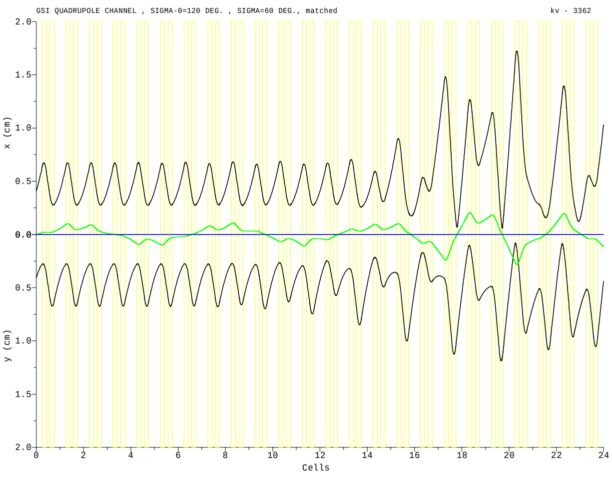

Exponential growth of the envelopes as long as $x, y \subset X_0, Y_0$

Quadrupole channel at $\sigma_0 = 120°$, $\sigma = 60°$: mismatch modes are unstable

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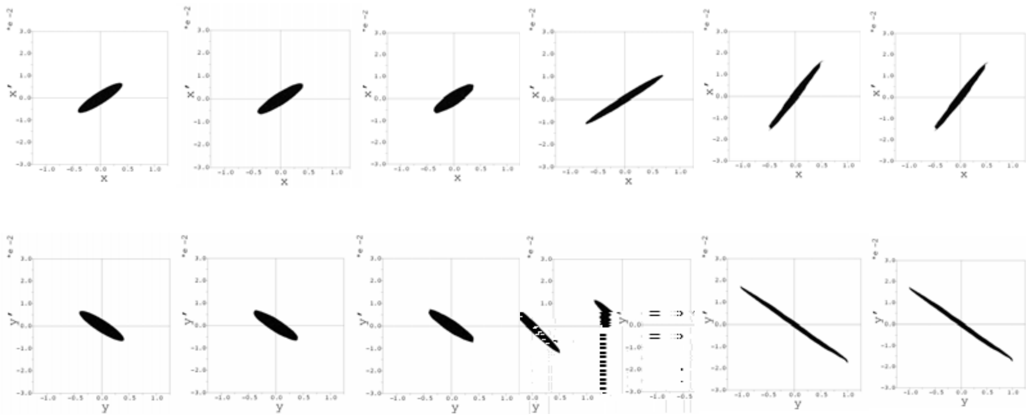

Cell 0 5 10 15 20 25

PIC simulation (K-V) of a quadrupole channel at $\sigma_0 = 120°$, $\sigma = 60°$

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6. Beam optics with space charge
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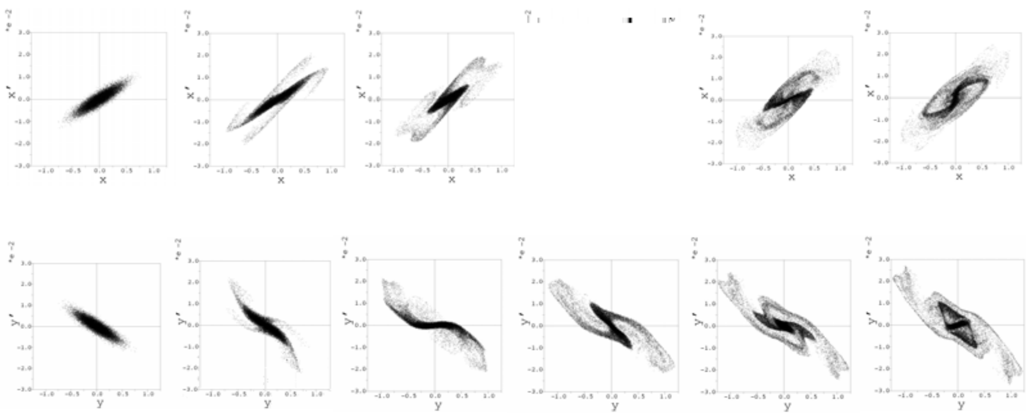

Cell 0 5 10 15 20 25

PIC simulation (Gauss) of a quadrupole channel at $\sigma_0 = 120°$, $\sigma = 60°$

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→ If the condition $x,y \subset X_0,Y_0$ is no longer satisfied, then the *linear* description of the envelope perturbations x,y is no longer valid. The growth of the perturbation is then no longer exponential.

→ Transition to a *chaotic* time evolution of the envelope.

<u>Conclusion:</u> For *long periodic* focusing systems under strong space charge conditions, we must limit the focusing strength of the lattice to

$$\sigma_0 \leq 90°$$

in order to ensure the stability of mismatch oscillations of the beam envelope.

The next topic will be: how to compute matched solutions of the K-V envelope equations under space charge conditions?

6. Beam Optics with space charge

6.4. Beam matching under space charge conditions

Let us recall the coupled, non-linear system of K-V equations

$$\frac{d^2}{ds^2}X(s)+k_{x,\mathrm{ext}}^2(s)\,X-\frac{2K}{X+Y}-\frac{\varepsilon_x^2}{X^3}=0$$

$$\frac{d^2}{ds^2}Y(s)+k_{y,\mathrm{ext}}^2(s)\,Y-\frac{2K}{X+Y}-\frac{\varepsilon_y^2}{Y^3}=0$$

➔No *analytic* solutions $X(s)$, $Y(s)$ exist.

➔For (almost) periodic ($k_{x,y}(s+S) = k_{x,y}(s)$) structures, matching ensures best beam properties (i.e., no particle losses, emittance preservation).

➔ Matching means: if $k_x(s)$, $k_y(s)$ are S-periodic functions, find initial values $X(0)$, $X'(0)$, $Y(0)$, $Y'(0)$ so that the envelopes are also S-periodic:

$$X(s+S)=X(s),\;X'(s+S)=X'(s),\quad Y(s+S)=Y(s),\;Y'(s+S)=Y'(s)$$

6. Beam optics with space charge
6.4. Beam matching

We define the abbreviation

$$\mathfrak{X}(s) = \begin{pmatrix} X(s) \\ Y(s) \\ X'(s) \\ Y'(s) \end{pmatrix}$$

Integrating the K-V equations from $s = 0$ to $s = S$ means to perform the non-linear mapping

$$\mathfrak{X}(0) \overset{\mathfrak{M}}{\mapsto} \mathfrak{X}(S), \quad \mathfrak{X}(S) = \mathfrak{M}\,\mathfrak{X}(0)$$

➔ Beam matching thus means to find the particular $\mathfrak{X}_m(0)$ with

$$\mathfrak{X}_m(0) = \mathfrak{M}\mathfrak{X}_m(0) \quad \Leftrightarrow \quad (\mathfrak{M} - \mathfrak{id})\,\mathfrak{X}_m(0) = \vec{0}, \quad \mathfrak{id}: \text{identity}$$

6. Beam optics with space charge
6.4. Beam matching

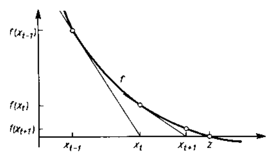
"Primitive" method to determine $\mathfrak{X}_m(0)$: we perform the iteration

$$\mathfrak{X}^{t+1}(0) = \frac{1}{2}\left(\mathfrak{X}^t(0) + \mathfrak{M}\mathfrak{X}^t(0)\right) = \frac{1}{2}\left(\mathfrak{X}^t(0) + \mathfrak{X}^t(S)\right)$$

This method works in many cases, but converges very slowly.

➔ <u>Better way:</u> Newton's method. For a function f of *one* variable x, we determine a zero x_0 of f via

$$x^{t+1} = x^t - \frac{f(x^t)}{f'(x^t)}$$



A *zero* x_0 of the function f is a *fixed point* of the function g, defined as

$$g(x) \overset{\text{Def}}{=} x - f(x), \quad f(x_0) = 0 \quad \Leftrightarrow \quad g(x_0) = x_0$$

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A *fixed point* of the function g is thus obtained by the iteration

$$x^{t+1} = x^t - \frac{g(x^t) - x^t}{g'(x^t) - 1}$$

In the case of functions of *multiple* variables, Newton's method is then

$$\mathfrak{X}^{t+1} = \mathfrak{X}^t - \left(J^t - E\right)^{-1}\left(\mathfrak{M}\mathfrak{X}^t - \mathfrak{X}^t\right)$$

with E denoting the *unit matrix*, and J the *Jacobi matrix* of $\mathfrak{M}$ at $\mathfrak{X}(S)$

Question: How do we determine the Jacobi matrix of a mapping that is only given as the *numerical* solution of a differential equation?

Answer: The *Jacobi matrix* is the *solution matrix* of the system of *linear perturbations* that is associated with the mapping.

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6. Beam optics with space charge
6.4. Beam matching
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➔ The *Jacobi matrix* of the K-V equations is the *solution matrix* of the system for the perturbations $x(s)$ and $y(s)$, i.e. the monodromy matrix

$$J = Z(S)$$

Recall: the matrix $Z(S)$ maps a *perturbation* $(x(0), y(0), x'(0), y'(0))$ through one period. Its initial value is the unit matrix: $Z(0) = E$.

➔ The iteration prescription to find a *fixed point* of the K-V equations, i.e., to determine the *matched beam parameters under space charge conditions* for a periodic structure thus finally writes:

$$\mathfrak{X}^{t+1}(0) = \mathfrak{X}^t(0) - \left(Z^t(S) - E\right)^{-1}\left(\mathfrak{X}^t(S) - \mathfrak{X}^t(0)\right)$$

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We quantify the residual deviation of the actual beam parameters from an accurate fixed point in terms of the dimensionless *mismatch factor*:

$$F_{\mathrm{MM}} = [\alpha(S) - \alpha(0)]^2 - [\beta(S) - \beta(0)][\gamma(S) - \gamma(0)]$$

with α, β, and γ denoting either the x,x' or the y,y' "ellipse" parameters that are related to the second beam moments.

Example: Quadrupole channel at $\sigma_0 = 90°$ with depressed tune $\sigma = 15°$. Initial values: zero current beam parameters (see Sec. 3).

Iteration cycle 1: Mismatch factor x,x′ = 1.9913E+01, Mismatch factor y,y′ = 1.4785E+01
Iteration cycle 2: Mismatch factor x,x′ = 9.3624E-01 , Mismatch factor y,y′ = 3.6228E+00
Iteration cycle 3: Mismatch factor x,x′ = 4.9410E-03 , Mismatch factor y,y′ = 6.5974E-03
Iteration cycle 4: Mismatch factor x,x′ = 3.0557E-09 , Mismatch factor y,y′ = 2.0786E-09
Iteration cycle 5: Mismatch factor x,x′ = 4.6515E-20 , Mismatch factor y,y′ = 1.7896E-20

7. Emittance growth effects

7.1. Emittance and field energy relations

In Sec. 6.2, we defined the "RMS emittance" as a characteristic beam property

$$\tilde{\varepsilon}_x^2(s) \stackrel{\text{Def}}{=} \overline{x^2}\,\overline{x'^2} - \left(\overline{xx'}\right)^2 \equiv \frac{1}{2N^2}\sum_{i,j=1}^{N}\left(x_i x'_j - x_j x'_i\right)^2$$

Of particular interest for the dynamics of charged particle beams are effects that cause a *change* (mostly a *growth*) of the RMS emittance.

In order to study these effects, we calculated its *s*-derivative:

$$\frac{d}{ds}\tilde{\varepsilon}_x^2(s) = \frac{2e}{mc^2\beta^2\gamma^3}\left(\overline{x^2}\;\overline{x'E_x} - \overline{xx'}\;\overline{xE_x}\right)$$

➔ The RMS emittance is constant for *linear* or *vanishing* E fields.

Question: what is the physical interpretation of this equation?

Continuous description: $\overline{x'E_x}$ is defined as:

$$\overline{x'E_x} \overset{\text{Def}}{=} \iiint E_x(x,y,z,s)\left[\iiint x' f(x,y,z,x',y',z',s)\,dx'dy'dz'\right] dxdydz$$

The expression in brackets is easily identified as a *local current density* in *x*-direction

$$j_x(x,y,z,s) = Nec\beta \iiint x' f(x,y,z,x',y',z',s)\,dx'dy'dz'$$

In terms of j_x, the moment $\overline{x'E_x}$ is obtained as:

$$\overline{x'E_x} = \frac{1}{Nec\beta} \iiint j_x E_x(x,y,z,s)\,dxdydz$$

$$\Rightarrow \quad Nec\beta\left(\overline{x'E_x} + \overline{y'E_y} + \overline{z'E_z}\right) = \iiint_V \vec{j}\cdot\vec{E}\,dV$$

This is the total rate of work (energy/time) that is done on the charges due to the EM fields, since the *B*-fields do no work.

The *energy conservation law* for charges in an EM field is given by Poynting's theorem:

$$\iiint_V \vec{j}\cdot\vec{E}\,dV + c\beta\frac{dW}{ds} + \oiint_S \vec{\mathfrak{S}}\,d\vec{S} = 0$$

This theorem follows directly from Maxwell's equations.

where *S* denotes the closed surface that bounds the integration volume *V*, and *W* stands for the electromagnetic field energy (here: in vacuum)

$$W = \frac{1}{2}\iiint_V \left(\varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) dV$$

The surface integral is constituted by the Poynting vector

$$\mathfrak{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

which provides the *energy flux* across the surface *S*.

➔ The surface integral

$$\oiint_S \vec{\mathfrak{S}}\, d\vec{S}$$

measures the total power (energy per time) that "escapes" through S.

➔ If we choose the integration volume V large enough to contain all charges, we can neglect this integral.

We further neglect the *magnetic* contribution to the field energy, so that

$$W = \frac{\varepsilon_0}{2} \iiint_V \vec{E}^2 dV$$

The moment equation then simplifies to:

$$\overline{x'E_x} + \overline{y'E_y} + \overline{z'E_z} = -\frac{\varepsilon_0}{2Ne}\frac{d}{ds}\iiint_V \vec{E}^2 dV = -\frac{1}{Ne}\frac{dW}{ds}$$

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7. Emittance growth effects
7.1. Emittance and field energy relations
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Particle description: $\overline{x'E_x}$ is defined as:

$$\overline{x'E_x} \overset{\text{Def}}{=} \frac{1}{N}\sum_{i=1}^{N} x_i' E_x(x_i, y_i, z_i, s)$$

with the E_x field derived from a space charge potential Φ_{sc} :

$$\Phi_{sc}(x_i, y_i, z_i, s) = \frac{e}{\varepsilon_0}\sum_{j\neq i}\frac{1}{\left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{1/2}},$$

$$E_x(x_i, y_i, z_i, s) = \frac{e}{\varepsilon_0}\sum_{j\neq i}\frac{x_i - x_j}{\left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{3/2}}$$

The electric field energy W is given by half the sum over all potentials Φ_{sc} :

$$W(s) = \frac{e^2}{2\varepsilon_0}\sum_i\sum_{j\neq i}\frac{1}{\left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{1/2}}$$

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$$\frac{dW(s)}{ds} = -\frac{e^2}{2\varepsilon_0} \sum_i \sum_{j\neq i} \frac{(x_i - x_j)(x_i' - x_j') + (y_i - y_j)(y_i' - y_j') + (z_i - z_j)(z_i' - z_j')}{\left[\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2 + \left(z_i - z_j\right)^2\right]^{3/2}}$$

$$\frac{e^2}{2\varepsilon_0} \quad \frac{x_i'(x_i - x_j) + y_i'(y_i - y_j) \quad z_i'(z_i \quad z_j) +}{\left(x_i \quad x_j\right)^2 \quad \left(y_i \quad y_j\right)^2 \quad \left(z_i \quad z_j\right)^2 {}^{3/2}} \quad x_j'(x_j \quad x_i) \quad y_j'(y_j \quad y_i) \quad z_j'(z_j \quad z_i)$$

➔ <u>We observe:</u> the result agrees with the previously obtained result from the *continuous* description.

Calculating the E-fields of different model charge distributions (next section!), we encounter a correction factor λ_3 that is close to unity (F.J. Sacherer 1971, I. Hofmann 1987):

$$\frac{\overline{xx'}}{\overline{x^2}}\overline{xE_x}+\frac{\overline{yy'}}{\overline{y^2}}\overline{yE_y}+\frac{\overline{zz'}}{\overline{z^2}}\overline{zE_z}=-\frac{\lambda_3}{Ne}\frac{dW_u}{ds}$$

$$\frac{1}{\overline{x^2}}\frac{d\tilde{\varepsilon}_x^2}{ds}+\frac{1}{\overline{y^2}}\frac{d\tilde{\varepsilon}_y^2}{ds}+\frac{1}{\overline{z^2}}\frac{d\tilde{\varepsilon}_z^2}{ds}=-\frac{2}{Nmc^2\beta^2\gamma^3}\left(\frac{dW}{ds}-\lambda_3\frac{dW_u}{ds}\right)$$

We now calculate the variation of F with respect to a variation of n:

$$\delta F = \iiint\limits_V \left[\varepsilon_0 \vec{E}\,\delta\vec{E} + N^{-1}\left(\lambda_x x^2 + \lambda_y y^2 + \lambda_z z^2\right)\delta n\right] dV \overset{!}{=} 0$$

The variation of the E field is related to the variation of n via Poisson's equation:

$$\nabla \delta\vec{E} = \frac{e}{\varepsilon_0}\delta n(x,y,z)$$

Partial integration of the E-related terms yields:

$$\delta F = \iiint\limits_V \left[e\Phi_{sc} + N^{-1}\left(\lambda_x x^2 + \lambda_y y^2 + \lambda_z z^2\right)\right]\delta n\, dV - \varepsilon_0 \iint\limits_S \Phi_{sc}\delta\vec{E}d\vec{S} \overset{!}{=} 0$$

with S denoting the closed surface around the volume V. As V can be chosen arbitrary large, we can neglect the surface integral over S.

The variation of n is related to variation of the position vector x by:

$$\delta n(x,y,z) = n(x+\delta x, y+\delta y, z+\delta z) - n(x,y,z) = \nabla n(x,y,z)\cdot\delta\vec{x}, \quad \delta\vec{x} = (\delta x, \delta y, \delta z)$$

Thus

$$\delta F = \iiint\limits_V \left[e\Phi_{sc} + N^{-1}\left(\lambda_x x^2 + \lambda_y y^2 + \lambda_z z^2\right)\right]\nabla n \cdot \delta\vec{x}\, dV \overset{!}{=} 0$$

Since the variation of the position vector is arbitrary, we get $\delta F = 0$ if

$$\delta F = 0 \quad \Leftrightarrow \quad \begin{cases} \nabla n = 0 & \Leftrightarrow \quad \text{uniform density} \\ Ne\Phi_{sc} = -(\lambda_x x^2 + \lambda_y y^2 + \lambda_z z^2) & \Leftrightarrow \quad \text{quadratic space charge potential} \end{cases}$$

Because of

$$\nabla^2\Phi_{sc} = -\frac{e}{\varepsilon_0}n(x,y,z)$$

we easily convince ourselves that the quadratic potential corresponds to a uniform density $n(x,y,z) = \text{const.}$

7. Emittance growth effects
7.1. Emittance and field energy relations

Coasting (unbunched) beam: we know from Sec. 6.2 that the moments xE_x and yE_y do *not* depend on the particular density profile as long as the transverse density $n(x,y,s)$ has *elliptic symmetry*:

$$\overline{xE_x}(s) = \frac{I}{4\pi\varepsilon_0 c\beta}\frac{\tilde{x}}{\tilde{x}+\tilde{y}}, \quad \overline{yE_y}(s) = \frac{I}{4\pi\varepsilon_0 c\beta}\frac{\tilde{y}}{\tilde{x}+\tilde{y}} \quad \Leftrightarrow \quad n(x,y,s) = n\left(\frac{x^2}{\tilde{x}^2(s)} + \frac{y^2}{\tilde{y}^2(s)}\right)$$

Thus

$$\frac{dW_u}{ds} = -Ne\left(\frac{\overline{xx'}}{\overline{x^2}}\,\overline{xE_x} + \frac{\overline{yy'}}{\overline{y^2}}\,\overline{yE_y}\right)$$

holds *for all elliptic density profiles* and no correction factor λ_2 occurs.

➔ The emittance-field-energy relation for unbunched beams follows as:

$$\frac{1}{\overline{x^2}}\frac{d\tilde{\varepsilon}_x^2}{ds} + \frac{1}{\overline{y^2}}\frac{d\tilde{\varepsilon}_y^2}{ds} = -\frac{2}{Nmc^2\beta^2\gamma^3}\frac{d}{ds}\left(W - W_u\right)$$

In this formula, we can conceive N as the number of particles *per unit length*. Then W, W_u are field energies *per unit length*!

7. Emittance growth effects
7.1. Emittance and field energy relations

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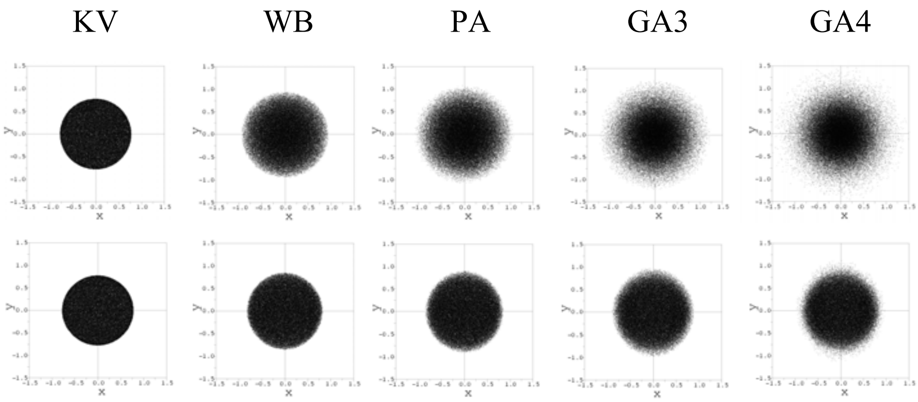
7. Emittance growth effects
7.2. Phase space probability densities



Transverse real space densities initially (upper row) and after one cell (lower row) in a quadrupole channel at σ_0=60°, σ=15°.

7. Emittance growth effects

7.2. Pf1

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For densities with *elliptical symmetry*, we define dimensionless ellipsoidal coordinates r_4, r_2, and ρ by:

$$a_x = x_{\max}, \quad a_y = y_{\max}, \quad a_{x'} = x'_{\max}, \quad a_{y'} = y'_{\max}$$

$$\hat{x} = x / a_x, \quad \hat{y} = y / a_y, \quad \hat{x}' = x' / a_{x'}, \quad \hat{y}' = y' / a_{y'}$$

$$r_4^2 = \hat{x}^2 + \hat{y}^2 + \hat{x}'^2 + \hat{y}'^2, \quad r_2^2 = \hat{x}^2 + \hat{y}^2, \quad \rho^2 = \hat{x}'^2 + \hat{y}'^2$$

The density $n(x,y)$ in real space is obtained by integrating f over x' and y':

$$n(x,y) = 4a_{x'}a_{y'} \int_{\hat{y}'=0}^{\sqrt{1-\hat{x}^2-\hat{y}^2}} \int_{\hat{x}'=0}^{\sqrt{1-\hat{x}^2-\hat{y}^2-\hat{y}'^2}} f(r_4^2)\, d\hat{x}' d\hat{y}' = \pi a_{x'} a_{y'} \int_{\rho^2=0}^{1-r_2^2} f(r_2^2+\rho^2)\, d(\rho^2)$$

The ratios $\overline{x^2} / a_x^2$, $\overline{x^4} / \overline{x^2}^2$ are then calculated from n via $(\hat{x} = r_2 \cos\varphi)$:

$$\overline{x^2} = 4a_x^3 a_y \int_{\hat{x}=0}^{1} \int_{\hat{y}=0}^{\sqrt{1-\hat{x}^2}} \hat{x}^2 n(\hat{x},\hat{y})\, d\hat{x} d\hat{y} = \frac{\pi a_x^3 a_y}{2} \int_0^1 r_2^2 n(r_2)\, d(r_2^2)$$

$$\overline{x^4} = 4a_x^5 a_y \int_{\hat{x}=0}^{1} \int_{\hat{y}=0}^{\sqrt{1-\hat{x}^2}} \hat{x}^4 n(\hat{x},\hat{y})\, d\hat{x} d\hat{y} = \frac{3\pi a_x^5 a_y}{8} \int_0^1 r_2^4 n(r_2)\, d(r_2^2)$$

The electric field in the *interior* of an unbunched beam with *elliptic symmetry* is calculated from the integral representation:

$$E_x^{\mathrm{i}}(x,y) = \frac{Nea_x a_y\, x}{2\varepsilon_0} \int_0^\infty \frac{n\left(\dfrac{x^2}{a_x^2+s} + \dfrac{y^2}{a_y^2+s}\right)}{\left(a_x^2+s\right)^{3/2}\left(a_y^2+s\right)^{1/2}} ds$$

The electric field in the *exterior* region of an unbunched beam with *circular symmetry* is the same for all distributions:

$$E_x^{\mathrm{e}}(x,y) = \frac{I}{2\pi\varepsilon_0 c\beta} \frac{1}{\sqrt{x^2+y^2}}$$

The field energy W per unit length is then the sum of the integrals over $dxdy$ of the interior and the exterior E^2 terms:

$$W = \frac{\varepsilon_0}{2} \iint (\vec{E}^{\mathrm{i}})^2 \, dxdy + \frac{\varepsilon_0}{2} \iint (\vec{E}^{\mathrm{e}})^2 \, dxdy$$

➔ For unbunched beams, we do *not* integrate over z. Therefore, W is the field energy *per unit length* and N the number of particles *per unit length*.

For a circular symmetric beam ($a_x = a_y = a$), we can write this as:

$$W = \frac{\pi\varepsilon_0}{2} a^2 \int_0^1 (E_r^{\mathrm{i}})^2 \, d\left(\frac{r^2}{a^2}\right) + w_0 \ln\frac{R}{a} \qquad [\text{Joules/meter}]$$

Herein, R denotes the radius of the beam pipe.

1. The KV distribution is defined as:

$$f_{\mathrm{KV}}(r_4) = \frac{1}{\pi^2 a_x a_y a_{x'} a_{y'}} \, \delta(r_4^2 - 1)$$

This is a *uniform* population of the hyper-sphere $r_4^2 = 1$.

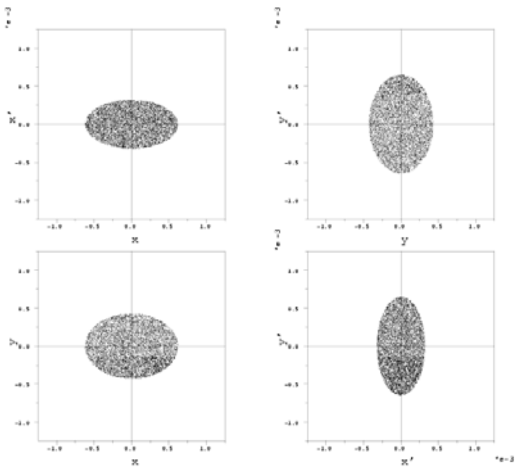
with δ the Dirac δ-function.

The density $n_{\mathrm{KV}}(x,y)$ in real space is obtained by integrating over x' and y':

$$n_{\mathrm{KV}}(x,y) = \frac{1}{\pi^2 a_x a_y a_{x'} a_{y'}} \, \pi a_{x'} a_{y'} \int_{\rho^2=0}^{1-r_2^2} \delta\left(\rho^2 - (1 - r_2^2)\right) d(\rho^2)$$

$$= \frac{1}{\pi a_x a_y} = \text{const.}$$

➔ Because of the symmetry of its coordinate dependence, the KV distribution leads to homogeneous densities in *all* 2-dimensional sub-planes.



(x,y), (x,x'), (y,y'), and (x',y') projections of the KV distribution.

7. Emittance growth effects
7.2. Phase space probability densities

The ratios $\overline{x^2}/a_x^2$, $\overline{x^4}/\overline{x^2}^2$ follow from:

$$\overline{x^2} = \frac{\pi a_x^3 a_y}{2}\,\frac{1}{\pi a_x a_y}\int_0^1 r_2^2\, d(r_2^2) = \frac{a_x^2}{4}, \qquad \overline{x^4} = \frac{3\pi a_x^5 a_y}{8}\,\frac{1}{\pi a_x a_y}\int_0^1 r_2^4\, d(r_2^2) = \frac{a_x^4}{8} \quad \Rightarrow \quad \frac{\overline{x^4}}{\overline{x^2}^2} = 2$$

The electric field in the *interior* of the KV distribution is obtained as:

$$E^{\mathrm{i}}_{x,\mathrm{KV}}(x,y) = \frac{N e a_x a_y\, x}{2\varepsilon_0}\,\frac{1}{\pi a_x a_y}\int_0^\infty \frac{ds}{(a_x^2+s)^{3/2}(a_y^2+s)^{1/2}}$$
$$= \frac{N e\, x}{2\pi\varepsilon_0}\,\frac{2}{a_x(a_x+a_y)} = \frac{I}{\pi\varepsilon_0 c\beta}\,\frac{x}{a_x(a_x+a_y)}$$

➔ The electric fields in the *interior* of the KV distribution are *linear* functions of the spatial coordinates x and y. For a *round* beam, we get:

$$E^{\mathrm{i}}_{r,\mathrm{KV}}(r) = \frac{I}{2\pi\varepsilon_0 c\beta}\,\frac{r}{a^2}$$

For the KV distribution, the field energy W per unit length as the sum of the integrals of the interior and the exterior E^2 terms evaluates to:

$$W_{\mathrm{KV}} = \frac{I^2}{8\pi\varepsilon_0 c^2\beta^2}\int_0^1 \frac{r^2}{a^2}\, d\left(\frac{r^2}{a^2}\right) + w_0 \ln\frac{R}{a}$$

$$= w_0\left(\frac{1}{4} + \ln\frac{R}{a}\right) \qquad [\mathrm{Joules/meter}]$$

We want to compare the field energies of different distributions with same RMS size.

➔ we must replace the beam radius a by the corresponding RMS term:

$$W_{\mathrm{KV}} = w_0\left(\frac{1}{4} + \ln\frac{R}{2\sqrt{\overline{x^2}}}\right) = w_0\left(\ln\frac{R}{\sqrt{\overline{x^2}}} + \frac{1}{4} - \ln 2\right)$$

The number 1/4 – ln2 is a *geometric* quantity and is a KV characteristic.

7. Emittance growth effects
7.2. Phase space probability densities

2. The “waterbag” distribution (WB) is defined as:

$$f_{\mathrm{WB}}(r_4) = \frac{2}{\pi^2 a_x a_y a_{x'} a_{y'}}\,\theta(1 - r_4^2) \qquad \theta(x) \overset{\mathrm{Def}}{=} \begin{cases} 0 & \text{for } \; x < 0 \\ 1 & \text{for } \; x > 0 \end{cases}$$

with θ the Heaviside step function. The density $n_{\mathrm{WB}}(x,y)$ in real space is then:

$$n_{\mathrm{WB}}(x,y) = \frac{2}{\pi^2 a_x a_y a_{x'} a_{y'}}\,\pi a_{x'} a_{y'} \int_{\rho^2=0}^{1-r_2^2} d(\rho^2)$$

$$= \frac{2}{\pi a_x a_y}\left(1 - r_2^2\right) = \frac{2}{\pi a_x a_y}\left(1 - \frac{x^2}{a_x^2} - \frac{y^2}{a_y^2}\right)$$

➔ The transverse density is a *paraboloid*. For the moment ratios, we get:

$$\overline{x^2} = \frac{\pi a_x^3 a_y}{2}\,\frac{2}{\pi a_x a_y}\int_0^1 r_2^2\left(1 - r_2^2\right) d(r_2^2) = \frac{a_x^2}{6}, \qquad \overline{x^4} = \frac{a_x^4}{16} \quad \Rightarrow \quad \frac{\overline{x^4}}{\overline{x^2}^2} = \frac{9}{4}$$

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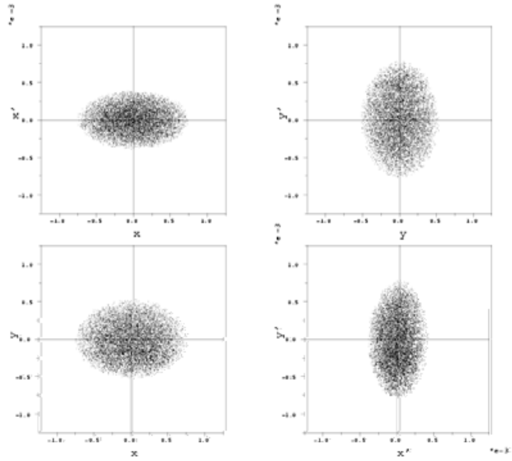

(x,y), (x,x'), (y,y'), and (x',y') projections of the WB distribution.

7. Emittance growth effects
7.2. Phase space probability densities

The electric field in the *interior* of the WB distribution follows as:

$$E^{\mathrm{i}}_{x,\mathrm{WB}}(x,y) = \frac{Ne\,x}{\pi\varepsilon_0}\int_0^\infty \frac{1-\dfrac{x^2}{a_x^2+s}-\dfrac{y^2}{a_y^2+s}}{\left(a_x^2+s\right)^{3/2}\left(a_y^2+s\right)^{1/2}}\,ds$$

$$= \frac{I}{\pi\varepsilon_0 c\beta}\,\frac{2x}{a_x(a_x+a_y)}\left[1-x^2\,\frac{2a_x+a_y}{3a_x^2(a_x+a_x)}-y^2\,\frac{1}{a_y(a_x+a_y)}\right]$$

For a round beam ($a_x = a_y = a$), this expression simplifies to:

$$E^{\mathrm{i}}_{x,\mathrm{WB}}(x,y) = \frac{I}{\pi\varepsilon_0 c\beta}\,\frac{x}{a^2}\left[1-\frac{x^2+y^2}{2a^2}\right]$$

$$E^{\mathrm{i}}_{r,\mathrm{WB}}(r) = \frac{I}{\pi\varepsilon_0 c\beta}\,\frac{r}{a^2}\left[1-\frac{r^2}{2a^2}\right]$$

The field energy W per unit length for the WB distribution is:

$$W_{\mathrm{WB}} = \frac{I^2}{4\pi\varepsilon_0 c^2\beta^2}\int_0^1 \frac{2r^2}{a^2}\left(1-\frac{r^2}{2a^2}\right)^2 d\left(\frac{r^2}{a^2}\right) + w_0 \ln\frac{R}{a}$$
$$= w_0\left(\frac{11}{24}+\ln\frac{R}{a}\right) \quad [\mathrm{Joules/meter}]$$

If we want to compare the field energies of different distributions with same RMS size, we must replace the beam radius a by the corresponding RMS term:

$$W_{\mathrm{WB}} = w_0\left(\frac{11}{24}+\ln\frac{R}{\sqrt{6\overline{x^2}}}\right) = w_0\left(\ln\frac{R}{\sqrt{\overline{x^2}}}+\frac{11}{24}-\frac{\ln 6}{2}\right)$$

The number 11/24 – (ln6)/2 is a WB characteristic.

7. Emittance growth effects
7.2. Phase space probability densities

3. The “parabolic” distribution (PA) is defined as:

$$f_{\mathrm{PA}}(r_4) = \frac{6}{\pi^2 a_x a_y a_{x'} a_{y'}}(1-r_4^2), \qquad 0 \le r_4^2 \le 1$$

The density $n_{\mathrm{PA}}(x,y)$ in real space is then:

$$n_{\mathrm{PA}}(x,y) = \frac{6}{\pi^2 a_x a_y a_{x'} a_{y'}}\,\pi a_{x'} a_{y'} \int_{\rho^2=0}^{1-r_2^2}\left(1-r_2^2-\rho^2\right)d(\rho^2)$$
$$= \frac{3}{\pi a_x a_y}\left(1-r_2^2\right)^2 = \frac{3}{\pi a_x a_y}\left(1-\frac{x^2}{a_x^2}-\frac{y^2}{a_y^2}\right)^2$$

For the PA distribution, the moment ratios evaluate to:

$$\overline{x^2} = \frac{\pi a_x^3 a_y}{2}\frac{3}{\pi a_x a_y}\int_0^1 r_2^2\left(1-r_2^2\right)^2 d(r_2^2) = \frac{a_x^2}{8}, \quad \overline{x^4} = \frac{3a_x^4}{80} \quad \Rightarrow \quad \frac{\overline{x^4}}{\overline{x^2}^2} = \frac{12}{5}$$

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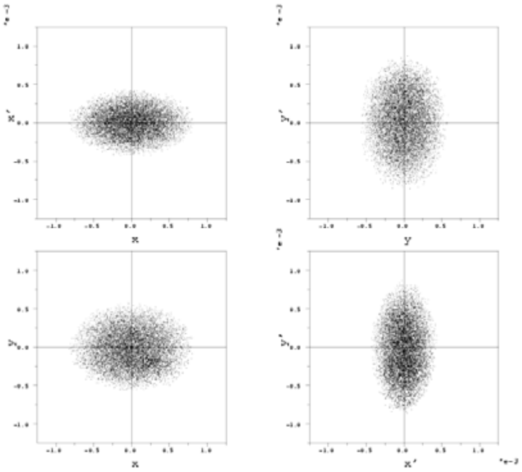


(x,y), (x,x'), (y,y'), and (x',y') projections of the PA distribution.

The electric field in the *interior* of the PA distribution follows as:

$$E^{\mathrm{i}}_{x,\mathrm{PA}}(x,y) = \frac{3Ne\,x}{2\pi\varepsilon_0} \int_0^\infty \frac{\left(1 - \frac{x^2}{a_x^2+s} - \frac{y^2}{a_y^2+s}\right)^2}{\left(a_x^2+s\right)^{3/2}\left(a_y^2+s\right)^{1/2}}\,ds$$

$$= \frac{I}{2\pi\varepsilon_0 c\beta}\frac{6x}{a_x(a_x+a_y)}\left[1 + \frac{1}{(a_x+a_y)^2}\left(x^4\frac{8a_x^2+9a_xa_y+3a_y^2}{15a_x^4} + 2x^2y^2\frac{3a_x+a_y}{3a_x^2a_y} + \right.\right.$$
$$\left.\left. y^4\frac{a_x+3a_y}{3a_y^3} - x^2\frac{4a_x^2+6a_xa_y+2a_y^2}{3a_x^2} - y^2\frac{2a_x+2a_y}{a_y}\right)\right]$$

For a round beam ($a_x = a_y = a$), this expression simplifies to:

$$E^{\mathrm{i}}_{x,\mathrm{PA}}(x,y) = \frac{I}{2\pi\varepsilon_0 c\beta}\frac{3x}{a^2}\left[1 - \frac{x^2+y^2}{a^2} + \frac{\left(x^2+y^2\right)^2}{3a^4}\right]$$

$$E^{\mathrm{i}}_{r,\mathrm{PA}}(r) = \frac{I}{2\pi\varepsilon_0 c\beta}\frac{3r}{a^2}\left[1 - \frac{r^2}{a^2} + \frac{r^4}{3a^4}\right]$$

The field energy W per unit length for the PA distribution is:

$$W_{\mathrm{PA}} = \frac{I^2}{4\pi\varepsilon_0 c^2\beta^2}\int_0^1 \frac{9r^2}{2a^2}\left(1-\frac{r^2}{a^2}+\frac{r^4}{3a^4}\right)^2 d\left(\frac{r^2}{a^2}\right)+w_0\ln\frac{R}{a}$$
$$= w_0\left(\frac{73}{120}+\ln\frac{R}{a}\right) \quad [\mathrm{Joules/meter}]$$

To compare the field energies of different distributions with same RMS size, we again replace the beam radius a by the corresponding RMS term:

$$W_{\mathrm{PA}} = w_0\left(\frac{73}{120}+\ln\frac{R}{\sqrt{8\overline{x^2}}}\right) = w_0\left(\ln\frac{R}{\sqrt{\overline{x^2}}}+\frac{73}{120}-\frac{1}{2}\ln 8\right)$$

The number 73/120 – (ln8)/2 is a PA characteristic.

4. The truncated “Gaussian” distribution (GA) is defined as:

$$f_{\mathrm{GA}}(r_4) = \frac{c_1}{4\pi^2 a_x a_y a_{x'} a_{y'}\alpha^4}\exp\left(-\frac{r_4^2}{2\alpha^2}\right), \quad 0\le r_4^2\le 1$$

with c_1 a correction factor to the normalization due to the truncation. According to the definition of r

$$f_{\mathrm{GA}}(x,y,x',y') = c_1\frac{1}{\sqrt{2\pi}a_x\alpha}\exp\left(-\frac{x^2}{2a_x^2\alpha^2}\right)\cdot\frac{1}{\sqrt{2\pi}a_y\alpha}\exp\left(-\frac{y^2}{2a_y^2\alpha^2}\right)\cdot\frac{1}{\sqrt{2\pi}a_{x'}\alpha}\exp\left(-\frac{x'^2}{2a_{x'}^2\alpha^2}\right)\cdot\frac{1}{\sqrt{2\pi}a_{y'}\alpha}\exp\left(-\frac{y'^2}{2a_{y'}^2\alpha^2}\right)$$

$$f_{\mathrm{GA}}(x,y,x',y') = \frac{1}{\sqrt{2\pi\overline{x^2}}}\exp\left(-\frac{x^2}{2\overline{x^2}}\right)\cdot\frac{1}{\sqrt{2\pi\overline{y^2}}}\exp\left(-\frac{y^2}{2\overline{y^2}}\right)\cdot\frac{1}{\sqrt{2\pi\overline{x'^2}}}\exp\left(-\frac{x'^2}{2\overline{x'^2}}\right)\cdot\frac{1}{\sqrt{2\pi\overline{y'^2}}}\exp\left(-\frac{y'^2}{2\overline{y'^2}}\right)$$

For the *truncated* GA distribution, the density $n_{GA}(x,y)$ in real space is:

$$n_{\mathrm{GA}}(x,y) = \frac{c_1}{2\pi a_x a_y \alpha^2} \int_0^{(1-r_2^2)/2\alpha^2} \exp\left(-\frac{r_2^2+\rho^2}{2\alpha^2}\right) d\left(\frac{\rho^2}{2\alpha^2}\right)$$
$$= \frac{c_1}{2\pi a_x a_y \alpha^2}\left[\exp\left(-\frac{\hat{x}^2+\hat{y}^2}{2\alpha^2}\right) - \exp\left(-\frac{1}{2\alpha^2}\right)\right]$$

We observe: $n_{GA} = 0$ if $x^2/a_x^2 + y^2/a_y^2 = 1$, as required.

The normalization factor c_1 is now obtained by integrating n_{GA} over x,y :

$$c_1^{-1} = \int_0^{1/2\alpha^2} \left[\exp\left(-\frac{r_2^2}{2\alpha^2}\right) - \exp\left(-\frac{1}{2\alpha^2}\right)\right] d\left(\frac{r_2^2}{2\alpha^2}\right)$$
$$= 1 - \left[1 + \frac{1}{2\alpha^2}\right]\exp\left(-\frac{1}{2\alpha^2}\right)$$

The normalized form of the truncated GA distribution finally writes:

$$f_{\mathrm{GA}}(r_4) = \frac{1}{4\pi^2 a_x a_y a_{x'} a_{y'} \alpha^4 \left[1 - \left[1 + \frac{1}{2\alpha^2}\right]\exp\left(-\frac{1}{2\alpha^2}\right)\right]} \exp\left(-\frac{r_4^2}{2\alpha^2}\right), \quad 0 \le r_4^2 \le 1$$

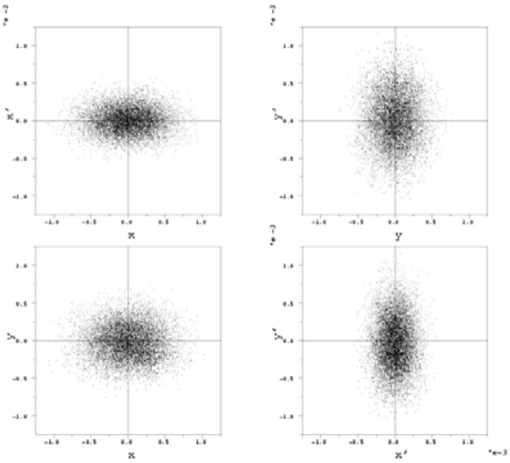
and, correspondingly, the density $n_{GA}(x,y)$:

$$n_{\mathrm{GA}}(x,y) = \frac{1}{2\pi a_x a_y \alpha^2 \left[1 - \left(1 + \frac{1}{2\alpha^2}\right)\exp\left(-\frac{1}{2\alpha^2}\right)\right]}\left[\exp\left(-\frac{1}{2\alpha^2}\left[\frac{x^2}{a_x^2} + \frac{y^2}{a_y^2}\right]\right) - \exp\left(-\frac{1}{2\alpha^2}\right)\right]$$

For a round beam ($a_x = a_y = a$), this expression simplifies to:

$$n_{\mathrm{GA}}(x,y) = \frac{1}{2\pi a^2 \alpha^2 \left[1 - \left(1 + \frac{1}{2\alpha^2}\right)\exp\left(-\frac{1}{2\alpha^2}\right)\right]}\left[\exp\left(-\frac{x^2+y^2}{2a^2\alpha^2}\right) - \exp\left(-\frac{1}{2\alpha^2}\right)\right]$$

7. Emittance growth effects
7.2. Phase space probability densities



(x,y), (x,x'), (y,y'), and (x',y') projections of the GA distribution.

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7. Emittance growth effects
7.2. Phase space probability densities

For the truncated GA distribution, the ratio $\overline{x^2}/a_x^2$ evaluates to:

$$\overline{x^2} = \frac{\pi a_x^3 a_y}{2} \frac{c_1 \alpha^2}{2\pi a_x a_y} \int_0^{1/\alpha^2} \frac{r_2^2}{\alpha^2}\left[\exp\left(-\frac{r_2^2}{2\alpha^2}\right) - \exp\left(-\frac{1}{2\alpha^2}\right)\right] d\left(\frac{r_2^2}{\alpha^2}\right)$$

$$= a_x^2 \alpha^2 \left(1 + \frac{1}{4\alpha^2} \frac{1}{1 + 2\alpha^2\left[1 - \exp\left(\frac{1}{2\alpha^2}\right)\right]}\right)$$

Thus

$$\alpha_\infty^2 = \frac{\overline{x^2}}{a_x^2} = \alpha^2 + \frac{1}{4 + 8\alpha^2\left(1 - \exp\left(\frac{1}{2\alpha^2}\right)\right)} \quad \underset{\alpha \to 0}{\longrightarrow} \quad \alpha^2$$

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The 4th moment is given by:

$$\overline{x^4} = \frac{3\pi a_x^5 a_y}{8}\frac{c_1}{2\pi a_x a_y \alpha^2}\int_0^1 r_2^4\left[\exp\left(-\frac{r_2^2}{2\alpha^2}\right)-\exp\left(-\frac{1}{2\alpha^2}\right)\right]d\left(r_2^2\right)$$
$$= 3c_1 a_x^4\alpha^4\left(1-\left[1+\frac{1}{2\alpha^2}+\frac{1}{8\alpha^4}+\frac{1}{48\alpha^6}\right]\exp\left(-\frac{1}{2\alpha^2}\right)\right)$$
$$\Rightarrow \frac{\overline{x^4}}{\overline{x^2}^2} \underset{\alpha\to 0}{\to} 3$$

The interior electric field is the solution of Poisson's equation:

$$E^{\mathrm{i}}_{r,\mathrm{GA}}(r) = \frac{I}{2\varepsilon_0 c\beta}\frac{1}{r}\int_0^{r^2} n_{\mathrm{GA}}(\tilde r)\,d(\tilde r^2)$$
$$= \frac{I}{4\pi\varepsilon_0 c\beta\alpha^2\left[1-\left(1+\frac{1}{2\alpha^2}\right)\exp\left(-\frac{1}{2\alpha^2}\right)\right]}\frac{1}{r}\int_0^{r^2/a^2}\left[\exp\left(-\frac{\tilde r^2}{2a^2\alpha^2}\right)-\exp\left(-\frac{1}{2\alpha^2}\right)\right]d\left(\frac{\tilde r^2}{a^2}\right)$$

For the *round* beam, we thus get an analytic solution for the electric field in the *interior* of the truncated GA distribution:

$$E^{\mathrm{i}}_{r,\mathrm{GA}}(r) = \frac{I}{2\pi\varepsilon_0 c\beta\left[1-\left(1+\frac{1}{2\alpha^2}\right)\exp\left(-\frac{1}{2\alpha^2}\right)\right]}\frac{1}{r}\left[1-\exp\left(-\frac{r^2}{2a^2\alpha^2}\right)-\frac{r^2}{2a^2\alpha^2}\exp\left(-\frac{1}{2\alpha^2}\right)\right]$$

We observe that the *internal* field E^{i} at $r = a$ indeed agrees with the general form for the *external* field E^{e} at $r = a$:

$$E^{\mathrm{i}}_{r,\mathrm{GA}}(a) = \frac{I}{2\pi\varepsilon_0 c\beta}\frac{1}{a}\frac{1-\exp\left(-\frac{1}{2\alpha^2}\right)-\frac{1}{2\alpha^2}\exp\left(-\frac{1}{2\alpha^2}\right)}{1-\left(1+\frac{1}{2\alpha^2}\right)\exp\left(-\frac{1}{2\alpha^2}\right)} = \frac{I}{2\pi\varepsilon_0 c\beta}\frac{1}{a} = E^{\mathrm{e}}_r(a)$$

➔ The requirement that the E field is *continuous* at the beam edge is fulfilled.

The field energy W per unit length for the GA distribution is then:

$$W_{\mathrm{GA}} = w_0 \frac{c_1^2}{2} \int_0^1 \frac{a^2}{r^2} \left[1 - \exp\left(-\frac{r^2}{2\alpha^2 a^2} \right) - \frac{r^2}{2\alpha^2 a^2} \exp\left(-\frac{1}{2\alpha^2} \right) \right]^2 d\left(\frac{r^2}{a^2} \right) + w_0 \ln \frac{R}{a}$$

$$= w_0 \left[\ln \frac{R}{\sqrt{\overline{x^2}}} + \ln \alpha_\infty + \frac{ \frac{C}{2} - \ln 2 - \ln \alpha + \left(1 - \frac{1}{2\alpha^2} \right) \exp\left(-\frac{1}{2\alpha^2} \right) - \left(1 - \frac{1}{16\alpha^4} \right) \exp\left(-\frac{1}{\alpha^2} \right) - \mathrm{Ei}\left(-\frac{1}{2\alpha^2} \right) + \frac{1}{2} \mathrm{Ei}\left(-\frac{1}{\alpha^2} \right) }{ \left(1 - \left[1 + \frac{1}{2\alpha^2} \right] \exp\left(-\frac{1}{2\alpha^2} \right) \right)^2 } \right]$$

$$\underset{\alpha \to 0}{\longrightarrow} w_0 \left[\ln \frac{R}{\sqrt{\overline{x^2}}} + \frac{C}{2} - \ln 2 \right] \approx w_0 \left[\ln \frac{R}{\sqrt{\overline{x^2}}} - 0.404539 \right]$$

Herein, C denotes Euler's constant, and Ei the integral exponential function

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^u}{u} du$$

Examples:

$$\alpha = \frac{1}{3} \quad \Rightarrow \quad W_{\mathrm{GA},3} \approx w_0 \left[\ln \frac{R}{\sqrt{\overline{x^2}}} - 0.416261 \right], \quad \frac{\overline{x^4}}{\overline{x^2}^2} = 2.7125$$

$$\alpha = \frac{1}{4} \quad \Rightarrow \quad W_{\mathrm{GA},4} \approx w_0 \left[\ln \frac{R}{\sqrt{\overline{x^2}}} - 0.406320 \right], \quad \frac{\overline{x^4}}{\overline{x^2}^2} = 2.9446$$

The upper limit of emittance growth is given if all "nonlinear field energy" $U(0)$ that is contained in the initial state of a distribution would be converted into kinetic energy.

➔ We must calculate $U(0) = W - W_{\mathrm{KV}}$ for the respective distribution:

$$U_{\mathrm{WB}}(0)/w_0 = (W_{\mathrm{WB}} - W_{\mathrm{KV}})/w_0 = \tfrac{11}{24} - \tfrac{1}{2}\ln 6 - \left(\tfrac{1}{4} - \ln 2\right) \approx 0.005601$$

$$U_{\mathrm{PA}}(0)/w_0 = (W_{\mathrm{PA}} - W_{\mathrm{KV}})/w_0 = \tfrac{73}{120} - \tfrac{1}{2}\ln 8 - \left(\tfrac{1}{4} - \ln 2\right) \approx 0.011760$$

$$U_{\mathrm{GA},3}(0)/w_0 = (W_{\mathrm{GA},3} - W_{\mathrm{KV}})/w_0 = -0.416261 - \left(\tfrac{1}{4} - \ln 2\right) \approx 0.026887$$

$$U_{\mathrm{GA},4}(0)/w_0 = (W_{\mathrm{GA},4} - W_{\mathrm{KV}})/w_0 = -0.406320 - \left(\tfrac{1}{4} - \ln 2\right) \approx 0.036827$$

The upper limit of the emittance growth is the calculated via:

$$\frac{\tilde{\varepsilon}_{x,\max}^2}{\tilde{\varepsilon}_x^2(0)} = 1 + \frac{2U(0)}{w_0} \left(\frac{\sigma_0^2}{\sigma^2} - 1 \right)$$

We take a typical quadrupole channel under high current load:

$$\sigma_0 = 60°, \quad \sigma = 15°$$

For these channel settings, upper limits for the emittance growth factors are:

$$\left.\frac{\tilde{\varepsilon}_{x,\max}}{\tilde{\varepsilon}_x(0)}\right|_{\mathrm{WB}} = \sqrt{1 + 0.011202\left(\frac{60^2}{15^2} - 1\right)} \approx 1.081$$

$$\left.\frac{\tilde{\varepsilon}_{x,\max}}{\tilde{\varepsilon}_x(0)}\right|_{\mathrm{PA}} = \sqrt{1 + 0.023520\left(\frac{60^2}{15^2} - 1\right)} \approx 1.163$$

$$\left.\frac{\tilde{\varepsilon}_{x,\max}}{\tilde{\varepsilon}_x(0)}\right|_{\mathrm{GA,3}} = \sqrt{1 + 0.053773\left(\frac{60^2}{15^2} - 1\right)} \approx 1.344$$

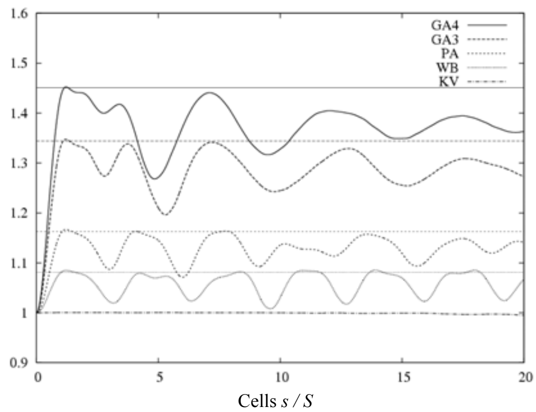
$$\left.\frac{\tilde{\varepsilon}_{x,\max}}{\tilde{\varepsilon}_x(0)}\right|_{\mathrm{GA,4}} = \sqrt{1 + 0.073654\left(\frac{60^2}{15^2} - 1\right)} \approx 1.451$$

It is now time to check these results by means of numerical simulations.

$\sigma_0 = 60°, \sigma = 15°$

$$\frac{\tilde{\varepsilon}_x^2(s)}{\tilde{\varepsilon}_x^2(0)}$$

The vertical line marks the location of the maximum emittance growth

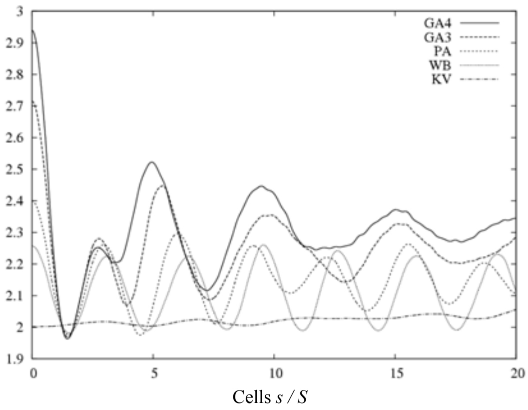


Emittance growth factors versus cell number for different initial distributions. The horizontal lines indicate the analytical results.

$\sigma_0 = 60°,\ \sigma = 15°$

$\dfrac{\overline{x^4}(s)}{\overline{x^2}^2(s)}$

The vertical line marks the location of the maximum emittance growth



Normalized 4th moments versus cell number for different initial distributions.

Problem: calculate

$$n(x,y),\quad \overline{x^2}\,/\,a_x^2,\quad \overline{x^4}\,/\,\overline{x^2}^2,$$
$$E_{x,y}(x,y),\quad W_{\mathrm{CO}},\quad (W_{\mathrm{CO}} - W_{\mathrm{KV}})\,/\,w_0$$

for the "conical" distribution

$$f_{\mathrm{CO}}(r_4) = \frac{10}{\pi^2 a_x a_y a_{x'} a_{y'}}(1 - r_4),\qquad 0 \le r_4 \le 1$$

and calculate the upper limit of the emittance growth factor due to the release of "nonlinear field energy" for a beam transport channel with

$$\sigma_0 = 60°,\qquad \sigma = 15°$$

Result:

$$\left.\frac{\tilde{\varepsilon}_{x,\max}}{\tilde{\varepsilon}_x(0)}\right|_{\mathrm{CO}} \approx 1.193$$

7. Emittance growth effects

7.3. Bunched beams, equipartitioning

We return to the general "emittance-field-energy relation" from Sec. 7.1.

$$\frac{1}{x^2}\frac{d\tilde{\varepsilon}_x^2}{ds}+\frac{1}{y^2}\frac{d\tilde{\varepsilon}_y^2}{ds}+\frac{1}{z^2}\frac{d\tilde{\varepsilon}_z^2}{ds}=-\frac{2}{Nmc^2\beta^2\gamma^3}\frac{d}{ds}\left(W-W_u\right)$$

Here, we applied the approximation $\lambda_3 = 1$.

This relation describes both

1. the rapid RMS emittance growth due a non-stationary initial phase-space distribution (➔ transient effect!),
2. the slow exchange of energies between the degrees of freedom due to a initially thermally imbalanced beam ("equipartitioning").

In the latter case, we will show that the field energy stays (almost) constant:

$$\frac{1}{x^2}\frac{d\tilde{\varepsilon}_x^2}{ds}+\frac{1}{y^2}\frac{d\tilde{\varepsilon}_y^2}{ds}+\frac{1}{z^2}\frac{d\tilde{\varepsilon}_z^2}{ds}\approx 0$$

This means that the total thermal energy is conserved during the temperature balancing process.

1. Proceeding as in Sec.7.2, we now calculate the moments x^2 and x^4 for a *uniformly* charged ellipsoid. Its charge density is:

$$\rho(x,y,z)=\rho_0=\frac{3Ne}{4\pi a_x a_y a_z}$$

Using scaled variables similar to Sec. 7.2, the second moment is given by:

$$\begin{aligned}\overline{x^2}&=\frac{3a_x^2}{4\pi}\int\limits_{r_3=0}^{1}\int\limits_{\vartheta=0}^{\pi}\int\limits_{\varphi=0}^{2\pi} r_3^2\sin^2\vartheta\cos^2\varphi\; r_3^2\sin\vartheta\; d\vartheta d\varphi dr_3\\&=\frac{3a_x^2}{4\pi}\int\limits_{r_3=0}^{1} r_3^4 dr\int\limits_{\vartheta=0}^{\pi}\sin^3\vartheta d\vartheta\int\limits_{\varphi=0}^{2\pi}\cos^2\varphi d\varphi=\frac{a_x^2}{5}\end{aligned}$$

In the same way, the fourth moment is obtained as:

$$\begin{aligned}\overline{x^4}&=\frac{3a_x^4}{4\pi}\int\limits_{r_3=0}^{1}\int\limits_{\vartheta=0}^{\pi}\int\limits_{\varphi=0}^{2\pi} r_3^4\sin^4\vartheta\cos^4\varphi\; r_3^2\sin\vartheta\; d\vartheta d\varphi dr_3\\&=\frac{3a_x^2}{4\pi}\int\limits_{r_3=0}^{1} r_3^6 dr\int\limits_{\vartheta=0}^{\pi}\sin^5\vartheta d\vartheta\int\limits_{\varphi=0}^{2\pi}\cos^4\varphi d\varphi=\frac{3a_x^4}{35},\quad\Rightarrow\frac{\overline{x^4}}{\overline{x^2}^2}=\frac{15}{7}\end{aligned}$$

Note: this ratio is different in the case of an unbunched beam.

To obtain analytic formulae, for the dynamics of equipartitioning, we restrict ourselves to *rotationally symmetric* ellipsoidal bunches.

The space charge potential ϕ_{sc} inside a *uniformly* populated ellipsoid is obtained from the general representation (dating back to Dirichlet):

$$\phi_{sc}^{i}(x,y,z)=\frac{a_x a_y a_z\rho_0}{4\varepsilon_0}\int\limits_0^\infty\frac{1-\frac{x^2}{a_x^2+s}-\frac{y^2}{a_y^2+s}-\frac{z^2}{a_z^2+s}}{\sqrt{(a_x^2+s)(a_y^2+s)(a_z^2+s)}}ds$$

Setting $a_x = a_y$, we find for the potential ϕ^{i} inside the ellipsoid:

$$\phi^{i}(x,y,z)=-\frac{\rho_0}{2\varepsilon_0}\left[(x^2+y^2)\frac{1-f}{2}+z^2 f\right],\qquad\rho_0=\frac{3Ne}{4\pi a_x^2 a_z}$$

with N the number of particles and f the ellipsoid's "form factor":

$$f = \begin{cases} \dfrac{1}{1-p^2} - \dfrac{p\cos^{-1} p}{\left(1-p^2\right)^{3/2}} & \text{for } p = \dfrac{a_z}{a_x} < 1 \\ \dfrac{1}{3} & \text{for p} = 1 \\ \dfrac{p\cosh^{-1} p}{\left(p^2-1\right)^{3/2}} - \dfrac{1}{p^2-1} & \text{for } p = \dfrac{a_z}{a_x} > 1 \end{cases}$$

The electric fields *inside* the ellipsoid follow as:

$$E_x^{\mathrm{i}} = \frac{\rho_0}{2\varepsilon_0}(1-f)\,x, \quad E_z^{\mathrm{i}} = \frac{\rho_0}{\varepsilon_0} f\,z, \quad \rho_0 = \frac{3Ne}{4\pi a_x^2 a_z}$$

Inserting the interior fields into the single particle equations of motion

$$x'' + k_{x,0}^2 x - \frac{eE_x^{\mathrm{i}}}{mc^2\beta^2\gamma^3} = 0, \quad z'' + k_{z,0}^2 z - \frac{eE_z^{\mathrm{i}}}{mc^2\beta^2\gamma^3} = 0$$

we find:

$$x'' + k_x^2 x = 0, \quad k_x^2 = k_{x,0}^2 - \frac{3(1-f)}{2a_x^2 a_z}\frac{Ne^2}{4\pi\varepsilon_0 mc^2\beta^2\gamma^3}$$

$$z'' + k_z^2 z = 0, \quad k_z^2 = k_{z,0}^2 - \frac{3f}{a_x^2 a_z}\frac{Ne^2}{4\pi\varepsilon_0 mc^2\beta^2\gamma^3}$$

➔All factors related to the number of particles can thus be expressed in terms of the “tunes” $k_{x,z}$ or, equivalently, phase advances $\sigma_{x,z}$:

$$\frac{Ne^2}{4\pi\varepsilon_0 mc^2\beta^2\gamma^3} = \frac{2a_x^2 a_z}{3(1-f)}\left(k_{x,0}^2 - k_x^2\right)$$

$$\frac{Ne^2}{4\pi\varepsilon_0 mc^2\beta^2\gamma^3} = \frac{a_x^2 a_z}{3f}\left(k_{z,0}^2 - k_z^2\right)$$

Note: this is *not* an approximation! We only replace the l.h.s. physical quantities by the r.h.s wave numbers of a *fictitious linear continuous system.* We do *not* claim that all particles of the real system oscillate with theses frequencies.

The total charge Ne inside the bunch can equivalently be expressed in terms of the *average* current I that propagates with the velocity $c\beta$ along a longitudinal distance L:

$$Ne = \frac{I \cdot L}{c\beta}$$

The tune relations now read with K the "generalized perveance" of Sec.6.1

$$k_x^2 = k_{x,0}^2 - \frac{3(1-f)}{2a_x^2 a_z}\frac{K}{2}L, \quad k_z^2 = k_{z,0}^2 - \frac{3f}{a_x^2 a_z}\frac{K}{2}L, \quad K = \frac{eI}{2\pi\varepsilon_0 mc^3\beta^3\gamma^3}$$

Setting $L = 2a_z$, then K is the *average scaled current* over the bunch length L. The tune relations now simplify to

$$k_x^2 = k_{x,0}^2 - K\frac{3(1-f)}{2a_x^2}, \quad k_z^2 = k_{z,0}^2 - K\frac{3f}{a_x^2}$$

For a spherical bunch (f = 1/3), the tune relations agree with the unbunched case of Sec. 7.1.

We need to calculate the electric field energy W_u^{i} inside a *uniformly* populated ellipsoid. We already know the electric fields, hence

$$W_u^{\mathrm{i}} = \frac{\rho_0^2\pi}{4\varepsilon_0}\int_{r^2=0}^{a_x^2}\int_{z=0}^{a_z\sqrt{1-r^2/a_x^2}}\left[(1-f)^2 r^2 + 4f^2z^2\right]d(r^2)\,dz, \quad \rho_0 = \frac{3Ne}{4\pi a_x^2 a_z}$$

The result is:

$$W_u^{\mathrm{i}} = \frac{N^2e^2}{4\pi\varepsilon_0}\frac{1}{10a_z}\left[\frac{3}{2}(1-f)^2 + \frac{3a_z^2}{a_x^2}f^2\right]$$

For a uniformly populated *spherical* bunch ($a_x = a_z = a$), i.e., for the form factor $f = 1/3$, this gives:

$$W_{u,s}^{\mathrm{i}} = \frac{N^2e^2}{4\pi\varepsilon_0}\frac{1}{10a}$$

The electric field outside the spherical ellipsoid is given by:

$$E_r^{\mathrm{o}}(r) = \frac{Ne}{4\pi\varepsilon_0}\frac{1}{r^2}$$

The related field energy is then:

$$W_{u,s}^{\mathrm{o}} = \frac{\varepsilon_0}{2}\frac{N^2e^2}{(4\pi\varepsilon_0)^2}4\pi\int_a^R \frac{1}{r^4}r^2 dr = \frac{N^2e^2}{4\pi\varepsilon_0}\left(\frac{1}{2a} - \frac{1}{2R}\right)$$

For the total field energy as the sum of W^{i} and W^{o}, we get the approximation:

$$W_u \cong \frac{N^2e^2}{4\pi\varepsilon_0}\left(\frac{9}{5a_z}\left[\frac{(1-f)^2}{2} + \frac{a_z^2}{a_x^2}f^2\right] - \frac{1}{2R}\right)$$

Here, we assumed that the external field has the symmetry of the ellipsoid's surface.

Will use this formula to calculate the corresponding W_u for the actual charge distribution with field energy W.

Under the assumption of approximately constant bunch size, the emittance-field-energy relation can be integrated to yield:

$$\frac{2}{\overline{x^2}}\Delta\tilde{\varepsilon}_x^2 + \frac{1}{\overline{z^2}}\Delta\tilde{\varepsilon}_z^2 = -\frac{2}{Nmc^2\beta^2\gamma^3}\Delta\left(W - W_u\right)$$

We want again to derive an expression for the relative emittance growth. To this end, we first rewrite the above equation in scaled quantities:

Note that only the *sum* of the emittance changes is obtained!

$$\begin{aligned} 2\frac{\Delta\tilde{\varepsilon}_x^2}{\overline{x^2}} + \frac{\Delta\tilde{\varepsilon}_z^2}{\overline{z^2}} &= -\frac{Ne^2}{2\pi\varepsilon_0 mc^2\beta^2\gamma^3}\frac{4\pi\varepsilon_0}{e^2N^2}\Delta\left(W - W_u\right), \quad Ne = \frac{I\cdot L}{c\beta} \\ &= -\frac{eI\cdot L}{2\pi\varepsilon_0 mc^3\beta^3\gamma^3}\frac{4\pi\varepsilon_0}{e^2N^2}\Delta\left(W - W_u\right) \\ &= -K\,\Delta\left(\frac{W - W_u}{W_L}\right), \quad W_L = \frac{N^2e^2}{4\pi\varepsilon_0 L} = \text{const.} \end{aligned}$$

with W_L a normalization field energy, and K the "generalized perveance".

Based on a thermodynamic reasoning, it makes sense to assume that the released field energy is *equally* distributed into the degrees of freedom

$$\frac{\Delta\tilde{\varepsilon}_x^2}{\overline{x^2}} = -\frac{K}{3}\Delta\left(\frac{W-W_u}{W_L}\right), \quad \frac{\Delta\tilde{\varepsilon}_z^2}{\overline{z^2}} = -\frac{K}{3}\Delta\left(\frac{W-W_u}{W_L}\right)$$

In terms of relative emittance changes, these equations write

$$\frac{\Delta\tilde{\varepsilon}_x^2}{\tilde{\varepsilon}_{x,0}^2} = -\frac{\overline{x^2}}{\tilde{\varepsilon}_{x,0}^2}\frac{K}{3}\Delta\left(\frac{W-W_u}{W_L}\right), \quad \frac{\Delta\tilde{\varepsilon}_z^2}{\tilde{\varepsilon}_{z,0}^2} = -\frac{\overline{z^2}}{\tilde{\varepsilon}_{z,0}^2}\frac{K}{3}\Delta\left(\frac{W-W_u}{W_L}\right)$$

or

$$\frac{\tilde{\varepsilon}_x^2(s)}{\tilde{\varepsilon}_x^2(0)} = 1 - \frac{K\overline{x^2}}{3\tilde{\varepsilon}_x^2(0)}\Delta\left(\frac{W-W_u}{W_L}\right), \quad \frac{\tilde{\varepsilon}_z^2(s)}{\tilde{\varepsilon}_z^2(0)} = 1 - \frac{K\overline{z^2}}{3\tilde{\varepsilon}_z^2(0)}\Delta\left(\frac{W-W_u}{W_L}\right)$$

The *form* of these equations agrees with the unbunched case

We will test by means of numerical simulations, to which degree these formulae are actually satisfied.

From the thermodynamic viewpoint, we observe that the equation

$$\frac{\Delta\tilde{\varepsilon}_x^2}{\tilde{\varepsilon}_{x,0}^2} = -\frac{\overline{x^2}}{\tilde{\varepsilon}_{x,0}^2}\frac{K}{3}\Delta\left(\frac{W-W_u}{W_L}\right)$$

can be rewritten be introducing the "temperature" T_x, the entropy S_x, and the "heat" Q

$$k_B T_x = \tfrac{1}{2} mc^2\beta^2 \frac{\tilde{\varepsilon}_x^2}{\overline{x^2}}, \quad S_x = k_B \ln\tilde{\varepsilon}_x, \quad Q = -mc^2\beta^2\frac{K}{12}\left(\frac{W-W_u}{W_L}\right)$$

as

$$\Delta S_x^{\mathrm{rev}} = \frac{\Delta Q^{\mathrm{rev}}}{T_x}$$

The change of the field energy divided by the actual temperature ("reduced heat") determines the entropy change.

We finally express the emittance-field-energy relation in terms of phase advances (tunes), making use of the already familiar relations:

$$\tilde{\varepsilon}_x^2 = k_x^2 \overline{x^2}^2, \qquad \tilde{\varepsilon}_z^2 = k_z^2 \overline{z^2}^2$$

$$k_x^2 = k_{x,0}^2 - \frac{3(1-f)}{4a_x^2}\frac{KL}{a_z}, \qquad k_z^2 = k_{z,0}^2 - \frac{3f}{2a_x^2}\frac{KL}{a_z}$$

and by replacing the normalization field energy W_L by $W_{u,s}^{\mathrm{i}}$ as the field energy *inside* a spherical bunch of the *same volume* as the actual bunch

$$W_{u,s}^{\mathrm{i}} = \frac{N^2 e^2}{4\pi\varepsilon_0}\frac{1}{10\sqrt[3]{a_x^2 a_z}} = W_L \frac{L}{10\sqrt[3]{a_x^2 a_z}} \neq \text{ const.}$$

We thus obtain:

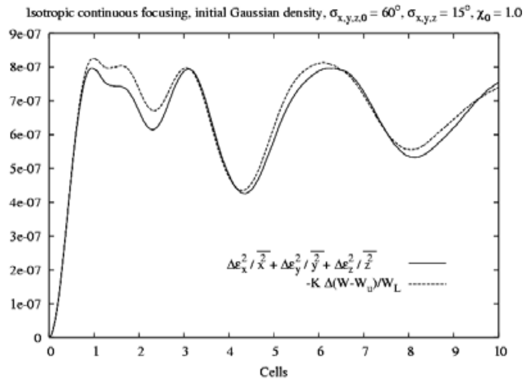
$$\begin{aligned}
\frac{\tilde{\varepsilon}_x^2(s)}{\tilde{\varepsilon}_x^2(0)} &= 1 - \frac{K\overline{x^2}}{3\tilde{\varepsilon}_{x,0}^2}\Delta\left(\frac{W - W_u}{W_L}\right) = 1 - \frac{\overline{x^2}}{30\tilde{\varepsilon}_{x,0}^2}\frac{KL}{\sqrt[3]{a_x^2 a_z}}\Delta\left(\frac{W - W_u}{W_{u,s}^{\mathrm{i}}}\right) \\
&= 1 - \frac{\overline{x^2}}{30\tilde{\varepsilon}_{x,0}^2}\left(k_{x,0}^2 - k_x^2\right)\frac{4a_x^2}{3(1-f)}\left(\frac{\overline{z^2}}{\overline{x^2}}\right)^{1/3}\Delta\left(\frac{W - W_u}{W_{u,s}^{\mathrm{i}}}\right) = 1 - \frac{1}{15\overline{x^2}}\left(\frac{k_{x,0}^2}{k_x^2} - 1\right)\frac{2a_x^2}{3(1-f)}\left(\frac{\overline{z^2}}{\overline{x^2}}\right)^{1/3}\Delta\left(\frac{W - W_u}{W_{u,s}^{\mathrm{i}}}\right) \\
&= 1 - \frac{1}{3}\left(\frac{k_{x,0}^2}{k_x^2} - 1\right)\frac{2/3}{1-f}\left(\frac{\overline{z^2}}{\overline{x^2}}\right)^{1/3}\Delta\left(\frac{W - W_u}{W_{u,s}^{\mathrm{i}}}\right) = 1 - \frac{1}{3}\left(\frac{\sigma_{x,0}^2}{\sigma_x^2} - 1\right)\frac{2/3}{1-f}\left(\frac{\overline{z^2}}{\overline{x^2}}\right)^{1/3}\Delta\left(\frac{W - W_u}{W_{u,s}^{\mathrm{i}}}\right)
\end{aligned}$$

Here, we made use of the fact that for a uniformly populated bunch, the RMS radii are related to the actual radii as:

$$a_x^2 = 5\overline{x^2}, \qquad a_z^2 = 5\overline{z^2}$$

For the longitudinal emittance change, we find in the same way:

7. Emittance growth effects
7.3. Bunched beams, equipartitioning



Numerical verification of the emittance-field energy equation for an initially equipartitioned bunch of Gaussian density in a *continuous* focusing channel.

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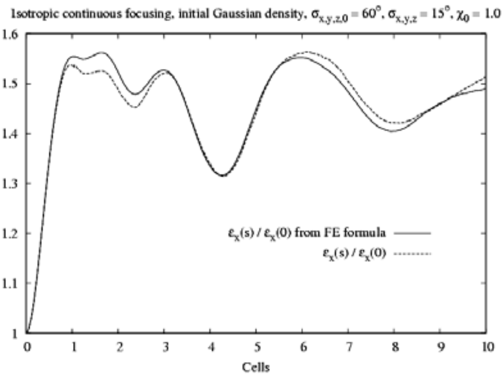
7. Emittance growth effects
7.3. Bunched beams, equipartitioning



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7. Emittance growth effects
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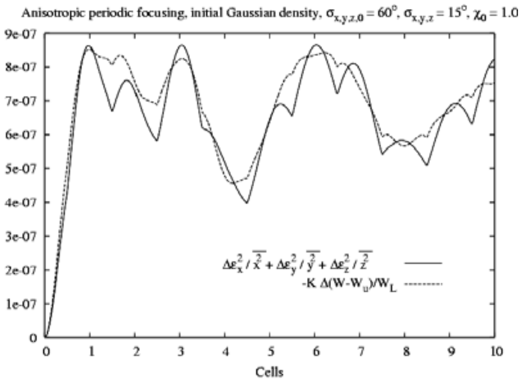

Numerical verification of the emittance-field energy equation for an initially equipartitioned bunch of Gaussian density in a *periodic* focusing channel.

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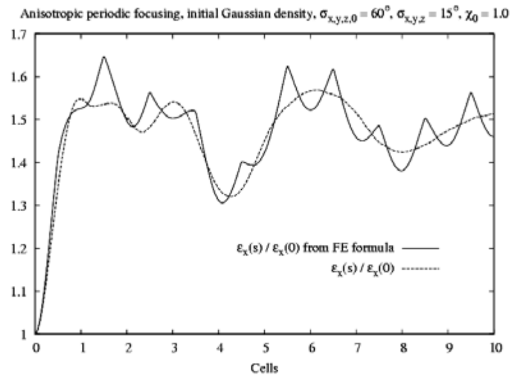
7. Emittance growth effects
7.3. Bunched beams, equipartitioning



Numerical verification of the emittance growth equation for an initially equipartitioned bunch of Gaussian density in a *periodic* focusing channel.

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2. The temperature balancing equations that maintain the total amount of thermal energy are:

$$k_B T_{x,0} = \tfrac{1}{2} mc^2\beta^2 \frac{\tilde{\varepsilon}_{x,0}^2}{\overline{x^2}}, \quad k_B T_{y,0} = \tfrac{1}{2} mc^2\beta^2 \frac{\tilde{\varepsilon}_{y,0}^2}{\overline{y^2}}, \quad k_B T_{z,0} = \tfrac{1}{2} mc^2\beta^2 \frac{\tilde{\varepsilon}_{z,0}^2}{\overline{z^2}}$$

$$k_B T_{\mathrm{eq}} = \tfrac{1}{2} mc^2\beta^2 \frac{1}{3}\left(\frac{\tilde{\varepsilon}_{x,0}^2}{\overline{x^2}} + \frac{\tilde{\varepsilon}_{y,0}^2}{\overline{y^2}} + \frac{\tilde{\varepsilon}_{z,0}^2}{\overline{z^2}}\right), \quad \tfrac{1}{2} mc^2\beta^2 \frac{\Delta\tilde{\varepsilon}_{x,y,z,f}^2}{\overline{x^2}} = k_B T_{x,y,z,0} - k_B T_{\mathrm{eq}}$$

$$\frac{\Delta\tilde{\varepsilon}_{x,f}^2}{\overline{x^2}} = -\frac{1}{3}\left(\frac{2\tilde{\varepsilon}_{x,0}^2}{\overline{x^2}} - \frac{\tilde{\varepsilon}_{y,0}^2}{\overline{y^2}} - \frac{\tilde{\varepsilon}_{z,0}^2}{\overline{z^2}}\right), \quad \frac{\Delta\tilde{\varepsilon}_{y,f}^2}{\overline{y^2}} = -\frac{1}{3}\left(\frac{2\tilde{\varepsilon}_{y,0}^2}{\overline{y^2}} - \frac{\tilde{\varepsilon}_{x,0}^2}{\overline{x^2}} - \frac{\tilde{\varepsilon}_{z,0}^2}{\overline{z^2}}\right), \quad \frac{\Delta\tilde{\varepsilon}_{z,f}^2}{\overline{z^2}} = -\frac{1}{3}\left(\frac{2\tilde{\varepsilon}_{z,0}^2}{\overline{z^2}} - \frac{\tilde{\varepsilon}_{x,0}^2}{\overline{x^2}} - \frac{\tilde{\varepsilon}_{y,0}^2}{\overline{y^2}}\right)$$

Obviously, the sum of these three equations is zero. ➔ The balancing equation satisfies the emittance equation for *constant* field energy.

For a round beam ($\varepsilon_x = \varepsilon_y$, $a_x = a_y$), the equations simplify to:

$$\frac{\Delta\tilde{\varepsilon}_{x,f}^2}{\overline{x^2}} = -\frac{1}{3}\left(\frac{\tilde{\varepsilon}_{x,0}^2}{\overline{x^2}} - \frac{\tilde{\varepsilon}_{z,0}^2}{\overline{z^2}}\right), \quad \frac{\Delta\tilde{\varepsilon}_{z,f}^2}{\overline{z^2}} = -\frac{2}{3}\left(\frac{\tilde{\varepsilon}_{z,0}^2}{\overline{z^2}} - \frac{\tilde{\varepsilon}_{x,0}^2}{\overline{x^2}}\right)$$

The relative emittance change is then:

$$\frac{\Delta\tilde{\varepsilon}_{x,f}^2}{\tilde{\varepsilon}_{x,0}^2} = -\frac{1}{3}\frac{\overline{x^2}}{\tilde{\varepsilon}_{x,0}^2}\left(\frac{\tilde{\varepsilon}_{x,0}^2}{\overline{x^2}} - \frac{\tilde{\varepsilon}_{z,0}^2}{\overline{z^2}}\right), \quad \frac{\Delta\tilde{\varepsilon}_{z,f}^2}{\tilde{\varepsilon}_{z,0}^2} = -\frac{2}{3}\frac{\overline{z^2}}{\tilde{\varepsilon}_{z,0}^2}\left(\frac{\tilde{\varepsilon}_{z,0}^2}{\overline{z^2}} - \frac{\tilde{\varepsilon}_{x,0}^2}{\overline{x^2}}\right)$$

We denote the temperature ratio, i.e. the ratio of the thermal energies by a parameter χ:

$$\chi = \frac{\tilde{\varepsilon}_z^2}{\overline{z^2}}\frac{\overline{x^2}}{\tilde{\varepsilon}_x^2}$$

With χ_0 the *initial* temperature imbalance, this means:

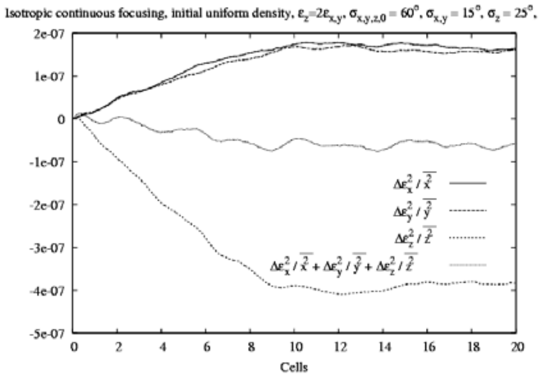
$$\frac{\Delta\tilde{\varepsilon}_{x,f}^2}{\tilde{\varepsilon}_{x,0}^2} = -\frac{1}{3}(1-\chi_0), \quad \frac{\Delta\tilde{\varepsilon}_{z,f}^2}{\tilde{\varepsilon}_{z,0}^2} = -\frac{2}{3}\left(1-\frac{1}{\chi_0}\right)$$

and hence:

$$\frac{\tilde{\varepsilon}_{x,f}^2}{\tilde{\varepsilon}_{x,0}^2} = \frac{1}{3}(2+\chi_0), \quad \frac{\tilde{\varepsilon}_{z,f}^2}{\tilde{\varepsilon}_{z,0}^2} = \frac{1}{3}\left(1+\frac{2}{\chi_0}\right)$$

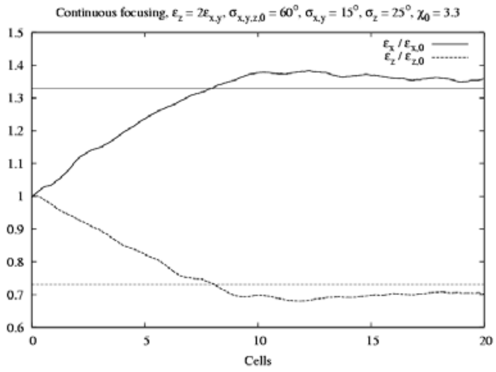
χ_0 has to be evaluated *after* the initial emittance growth due to an initial charge density mismatch

7. Emittance growth effects
7.3. Bunched beams, equipartitioning



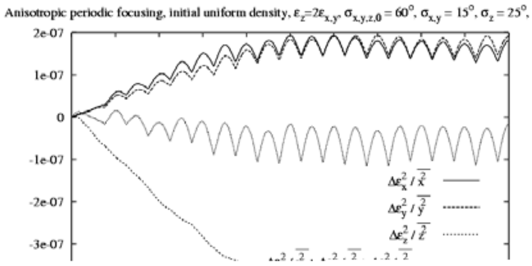
Transverse and longitudinal temperature changes versus cell number for an initially non-equipartitioned bunch of *uniform* density in a *continuous* focusing channel. The total temperature stays nearly constant.

7. Emittance growth effects
7.3. Bunched beams, equipartitioning



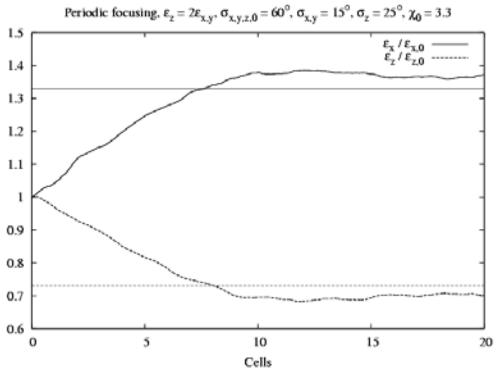
Transverse and longitudinal emittance growth factors versus cell number for an initially non-equipartitioned bunch of *uniform* density in a *continuous* focusing channel. The horizontal lines indicate the analytical growth estimates.

7. Emittance growth effects
7.3. Bunched beams, equipartitioning



Transverse and longitudinal temperature changes versus cell number for an initially non-equipartitioned bunch of *uniform* density in a *periodic* focusing channel. The total temperature stays nearly constant.

7. Emittance growth effects
7.3. Bunched beams, equipartitioning



Transverse and longitudinal emittance growth factors versus cell number for an initially non-equipartitioned bunch of *uniform* density in a *periodic* focusing channel. The horizontal lines indicate the analytical growth estimates.

8. Stationary phase-space distributions

8.1. General theory

Motivation for studying self-consistent phase-space probability density functions *f* :

➔Of course, we cannot actively "produce" self-consistent beam right off the ion source.

➔We can generate self-consistent distributions of particles in computer simulations and thus avoid transient emittance growth effects.

➔We thus learn about phase-space distributions of *real beams* that have adapted themselves to the external focusing and its self-fields.

➔We observe that real beams are quite different from the KV model. Plasma shielding effect: the particles arrange in a way that the *interior* of the beam is shielded from the external forces.

8. Stationary phase-space distributions
8.1. General theory

We have seen in Sec. 1 that in the limit of a *continuous* charge density – with corresponding *continuous* self-fields – the Liouville theorem applies for *all* phase-space probability density functions *f*:

$$\frac{df}{ds} = 0$$

In Sec. 7.2, we defined various density functions *f* on the basis of a geometrical reasoning.

➔Apart from the K-V distribution, all these distributions lead to emittance growth for non-zero current *I*. How do we avoid this?

We assume *continuous focusing forces* along *s*. Then, the condition for a phase-space probability density function *f* to be *stationary* is:

$$\frac{\partial f}{\partial s} = 0$$

This means that the density *f* is invariant at *all individual* phase-space locations.

8. Stationary phase-space distributions
8.1. General theory

In order to work out this condition for a stationary phase-space density, we set up the Hamiltonian of a phase-space point. For an unbunched round beam in a continuous focusing channel with tune k_0, we have:

$$H(x,p_x,y,p_y) = \frac{1}{2m\gamma}\left(p_x^2 + p_y^2\right) + \tfrac{1}{2} m\gamma c^2\beta^2 k_0^2\left(x^2 + y^2\right) + \frac{e}{\gamma^2} V_{sc}(x,y)$$

We easily verify that the subsequent canonical equations yield the correct equations of motion. Because of

$$s = c\beta t, \quad p_x = m\gamma\dot{x},\ p_y = m\gamma\dot{y}, \quad \ddot{x} = c^2\beta^2 x'',\ \ddot{y} = c^2\beta^2 y'', \quad E_x = -\frac{\partial V_{sc}}{\partial x},\ E_y = -\frac{\partial V_{sc}}{\partial y}$$

the equations of motion write in "trace-space" notation:

$$x'' + k_0^2 x - \frac{e}{mc^2\beta^2\gamma^3} E_x(x,y) = 0, \qquad y'' + k_0^2 y - \frac{e}{mc^2\beta^2\gamma^3} E_y(x,y) = 0$$

which can be derived from a "trace-space" Hamiltonian $\bar{H}(x,x',y,y')$ via:

$$x' = \frac{\partial \bar{H}(x,x',y,y')}{\partial x'}, \quad x'' = -\frac{\partial \bar{H}(x,x',y,y')}{\partial x}, \quad y' = \frac{\partial \bar{H}(x,x',y,y')}{\partial y'}, \quad y'' = -\frac{\partial \bar{H}(x,x',y,y')}{\partial y}$$

The normalized "trace-space" Hamiltonian is, therefore,

$$\bar{H}(x,x',y,y') = \tfrac{1}{2}\left(x'^2 + y'^2\right) + \tfrac{1}{2}k_0^2\left(x^2 + y^2\right) + \frac{e}{mc^2\beta^2\gamma^3}V_{sc}(x,y)$$

With this Hamiltonian, we may write Liouville's theorem as follows:

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial \bar{H}}{\partial x'} - \frac{\partial f}{\partial x'}\frac{\partial \bar{H}}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \bar{H}}{\partial y'} - \frac{\partial f}{\partial y'}\frac{\partial \bar{H}}{\partial y} = 0$$

The above condition for a stationary distribution thus means that *f* must commute with the Hamiltonian:

$$\frac{\partial f}{\partial s} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \frac{\partial f}{\partial x}\frac{\partial \bar{H}}{\partial x'} - \frac{\partial f}{\partial x'}\frac{\partial \bar{H}}{\partial x} \stackrel{!}{=} 0, \quad \frac{\partial f}{\partial y}\frac{\partial \bar{H}}{\partial y'} - \frac{\partial f}{\partial y'}\frac{\partial \bar{H}}{\partial y} \stackrel{!}{=} 0$$

This is obviously the case if f is a function of H only:

$$f = f(\bar{H}) \quad \Rightarrow \quad \frac{\partial f}{\partial \bar{H}}\frac{\partial \bar{H}}{\partial x}\frac{\partial \bar{H}}{\partial x'} - \frac{\partial f}{\partial \bar{H}}\frac{\partial \bar{H}}{\partial x'}\frac{\partial \bar{H}}{\partial x} = 0$$

This means that phase-space hyper-planes of constant energy *H* must have constant density *f*.

- ➔ To obtain stationary distributions, we must make sure that phase-space hyper-planes of *constant* energy ("Isohamiltonians") are endowed with a constant particle probability density *f*.
- ➔ "Self-consistency" problem: due to the SC potential, the particular shapes of the Isohamiltonians in turn depend on the probability *f*.
- ➔ We cannot define the probability density function *a priori* (as done before), but must calculate the particular shape of *f* in a *closed loop*.

To this end, we calculate the *maximum energy* that a particle can have within the beam. This is obviously the energy of a particle at the beam edge *a*:

$$\bar{H}_{max} = \tfrac{1}{2}k_0^2 a^2 + \frac{e}{mc^2\beta^2\gamma^3}V_{sc}(a), \quad a^2 = (x^2 + y^2)_{max} = r_{max}^2$$

At a later time, the maximum energy particle is located somewhere within the beam:

$$\bar{H}_{\max} = \tfrac{1}{2}\rho^2_{\max}(r) + \tfrac{1}{2}k_0^2 r^2 + \frac{e}{mc^2\beta^2\gamma^3}V_{sc}(r), \quad \rho^2_{\max}(r) = (x'^2 + y'^2)_{\max}(r)$$

This means for $\rho^2_{\max}(r)$:

$$\rho^2_{\max}(r) = k_0^2\left[a^2 - r^2\right] + \frac{2e}{mc^2\beta^2\gamma^3}\left[V_{sc}(a) - V_{sc}(r)\right]$$

Defining the normalized effective potential $V_{\mathrm{eff}}(r)$ as sum of the external and the space charge potential

$$V_{\mathrm{eff}}(r) = \tfrac{1}{2}k_0^2 r^2 + \frac{e}{mc^2\beta^2\gamma^3}V_{sc}(r) \quad \Rightarrow \quad \bar{H}_{\max} = V_{\mathrm{eff}}(a)$$

the expression for the square of the maximum angle writes:

$$\rho^2_{\max}(r) = 2\left[V_{\mathrm{eff}}(a) - V_{\mathrm{eff}}(r)\right] = 2\left[\bar{H}_{\max} - V_{\mathrm{eff}}(r)\right]$$

To calculate the particle density $n(r)$, we must integrate f over all possible velocities (resp. angles) up to the limit $\rho^2_{\max}(r)$:

$$n(r) = \pi \int_{\rho^2=0}^{\rho^2_{\max}(r)} f\left(\bar{H}(r,\rho)\right) d(\rho^2)$$

To obtain the related space-charge potential V_{sc}, we insert this density into the Poisson equation:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dV_{sc}}{dr}\right] = -\frac{\pi N e}{\varepsilon_0}\int_{\rho^2=0}^{\rho^2_{\max}(r)} f\left(\bar{H}(r,\rho)\right) d(\rho^2)$$

Replacing V_{sc} with the effective potential V_{eff}, we obtain equivalently:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dV_{\mathrm{eff}}}{dr}\right] = 2k_0^2 - \frac{\pi N e^2}{\varepsilon_0 mc^2\beta^2\gamma^3}\int_{\rho^2=0}^{\rho^2_{\max}(r)} f\left(\bar{H}(r,\rho)\right) d(\rho^2)$$

We furthermore change the formal integration variable:

$$\bar{H} = \tfrac{1}{2}\rho^2 + V_{\text{eff}}(r) \quad \Rightarrow \quad \rho^2 = 2\left(\bar{H} - V_{\text{eff}}(r)\right), \quad d(\rho^2) = 2d\bar{H},\ r = \text{const.}$$

This yields:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dV_{\text{eff}}(r)}{dr}\right] = 2k_0^2 - \frac{2\pi Ne^2}{\varepsilon_0 mc^2\beta^2\gamma^3}\int_{V_{\text{eff}}(r)}^{\bar{H}_{\max}} f(\bar{H})\,d\bar{H}$$

This form reflects the requirement that surfaces of constant energy must have constant density.

With N the number of particles per unit length (see Sec. 7.1), we have $Ne = I/c\beta$:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dV_{\text{eff}}(r)}{dr}\right] = 2k_0^2 - 4\pi^2 K\int_{V_{\text{eff}}(r)}^{\bar{H}_{\max}} f(\bar{H})\,d\bar{H}, \quad K = \frac{eI}{2\pi\varepsilon_0 mc^3\beta^3\gamma^3}$$

With F denoting an *integral function* of the given probability density function f, we finally obtain:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dV_{\text{eff}}(r)}{dr}\right] + 4\pi^2 K\left[F\left(V_{\text{eff}}(a)\right) - F\left(V_{\text{eff}}(r)\right)\right] = 2k_0^2$$

This is a closed equation that enables us to determine $V_{\text{eff}}(r)$ for given tune k_0, scaled current K, and density f with its integral F.

The boundary of the populated phase space is obtained from

$$\rho_{\max}^2(r) = 2\left[V_{\text{eff}}(a) - V_{\text{eff}}(r)\right] \quad \Rightarrow \quad \rho_{\max}^2(0) = 2\left[V_{\text{eff}}(a) - V_{\text{eff}}(0)\right]$$

hence

$$\frac{\rho_{\max}^2(r)}{\rho_{\max}^2(0)} = \frac{V_{\text{eff}}(a) - V_{\text{eff}}(r)}{V_{\text{eff}}(a) - V_{\text{eff}}(0)} = 1 - \frac{V_{\text{eff}}(r) - V_{\text{eff}}(0)}{V_{\text{eff}}(a) - V_{\text{eff}}(0)}$$

$$\Rightarrow \quad \frac{\rho_{\max}^2(r)}{\rho_{\max}^2(0)} + \frac{V_{\text{eff}}(r) - V_{\text{eff}}(0)}{V_{\text{eff}}(a) - V_{\text{eff}}(0)} = 1$$

➔ Only for a quadratic effective potential, we obtain an elliptic symmetry.

We shall investigate in the following the properties of "self-consistently" defined phase-space probability densities with examples of the KV, the WB, and the GA distributions.

8. Stationary phase-space distributions

8.2. KV distribution

In terms of a function of the energy H, the KV distribution is defined analogously to its "geometrical" definition as:

$$f_{KV}(\overline{H}) = \frac{1}{2\pi^2 a^2}\,\delta\left(\overline{H}_{\max} - \overline{H}\right)$$

Problem: verify the normalization factor in front of the δ-function.

We insert this definition into the equation for $V_{\text{eff}}(r)$:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dV_{\text{eff,KV}}(r)}{dr}\right] = 2k_0^2 - \frac{2K}{a^2}\int_{V_{\text{eff,KV}}(r)}^{\overline{H}_{\max}} \delta\left(\overline{H}_{\max} - \overline{H}\right)d\overline{H} = 2\left(k_0^2 - \frac{K}{a^2}\right)$$

The effective potential $V_{\text{eff}}(r)$ is immediately obtained as:

$$V_{\text{eff,KV}}(r) - V_{\text{eff,KV}}(0) = \tfrac{1}{2}k^2 r^2, \qquad k^2 = k_0^2 - \frac{K}{a^2}$$

We already encountered k in Sec 6.1, where we defined it as the *depressed tune*.

The space charge potential $V_{\text{sc,KV}}(r)$ follows from the effective potential $V_{\text{eff,KV}}(r)$ as:

$$V_{\text{sc,KV}}(r) - V_{\text{sc,KV}}(0) = -\frac{I}{4\pi\varepsilon_0 c\beta}\frac{r^2}{a^2}$$

and hence the electric space-charge field inside the beam

$$E^{\text{i}}_{r,\text{KV}}(r) = \frac{I}{2\pi\varepsilon_0 c\beta}\frac{r}{a^2}$$

We observe that this field agrees with "geometrical" definition of the KV distribution from Sec. 7.2.

➔ The "geometrical" and the "self-consistent" definitions of the KV distribution coincide.

8. Stationary phase-space distributions

8.3. "Waterbag" (WB) distribution

The “self-consistent” WB distribution is defined analogously to its “geometrical” definition in terms of the step function θ as:

$$f_{\mathrm{WB}}(\bar{H}) = c_0 \theta\left(\bar{H}_{\max} - \bar{H}\right)$$

This defines a *constant* density *f* within the self-consistent boundary.

with c_0 denoting a normalization constant (to be determined). The equation for the effective potential is now:

$$\frac{1}{r}\frac{d}{dr}\left[r \frac{dV_{\mathrm{eff,WB}}(r)}{dr}\right] = 2k_0^2 - 4\pi^2 K c_0 \int_{V_{\mathrm{eff,WB}}(r)}^{\bar{H}_{\max}} \theta\left(\bar{H}_{\max} - \bar{H}\right) d\bar{H}$$
$$= 2k_0^2 - 4\pi^2 K c_0 \left[V_{\mathrm{eff,WB}}(a) - V_{\mathrm{eff,WB}}(r)\right]$$

To work out the effective potential $V_{\mathrm{eff}}(r)$, we make the substitutions:

$$W(r) = \frac{2k_0^2}{\kappa^2} - V_{\mathrm{eff,WB}}(a) + V_{\mathrm{eff,WB}}(r), \quad \kappa^2 = 4\pi^2 K c_0$$

The equation for $W(r)$ is then:

$$\frac{1}{r}\frac{d}{dr}\left[r \frac{dW(r)}{dr}\right] = \kappa^2 W(r)$$

which has the form of a Bessel equation. Its only physically meaningful solution for our case is:

$$W(r) = W(0)\, I_0(\kappa r)$$

with I_0 denoting the zeroth modified Bessel function. Inserting $W(r)$, the effective potential is obtained as:

$$V_{\mathrm{eff,WB}}(a) - V_{\mathrm{eff,WB}}(r) = \frac{2k_0^2}{\kappa^2} - W(0)\, I_0(\kappa r),$$

$$\overset{r=a}{\Rightarrow} \quad \frac{2k_0^2}{\kappa^2} = W(0)\, I_0(\kappa a)$$

$$\Rightarrow \quad V_{\mathrm{eff,WB}}(a) - V_{\mathrm{eff,WB}}(r) = \frac{2k_0^2}{\kappa^2 I_0(\kappa a)}\left[I_0(\kappa a) - I_0(\kappa r)\right], \quad V_{\mathrm{eff,WB}}(r) - V_{\mathrm{eff,WB}}(0) = \frac{2k_0^2}{\kappa^2 I_0(\kappa a)}\left[I_0(\kappa r) - 1\right]$$

8. Stationary phase-space distributions
8.3. WB distribution

From the functional form of the effective potential

$$\frac{V_{\mathrm{eff,WB}}(r)-V_{\mathrm{eff,WB}}(0)}{V_{\mathrm{eff,WB}}(a)-V_{\mathrm{eff,WB}}(0)}=\frac{I_0(\kappa r)-1}{I_0(\kappa a)-1}$$

the self-consistent phase-space boundary is directly found:

$$\frac{\rho^2_{\max}(r)}{\rho^2_{\max}(0)}+\frac{I_0(\kappa r)-1}{I_0(\kappa a)-1}=1$$

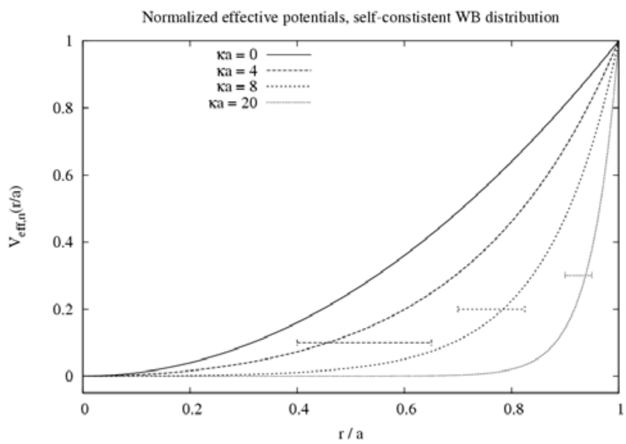
The self-consistent charge density $g_{\mathrm{WB}}(r)$ that is associated with the already calculated effective potential follows as:

$$g_{\mathrm{WB}}(r)=g_0\left[1-\frac{I_0(\kappa r)}{I_0(\kappa a)}\right],\qquad g_0=k_0^2\,\frac{2\varepsilon_0 mc^2\beta^2\gamma^3}{e}$$

The “geometrical” distributions of Sec.7.2 exceed the density g_0 ➔emittance growth due to release of free field energy.

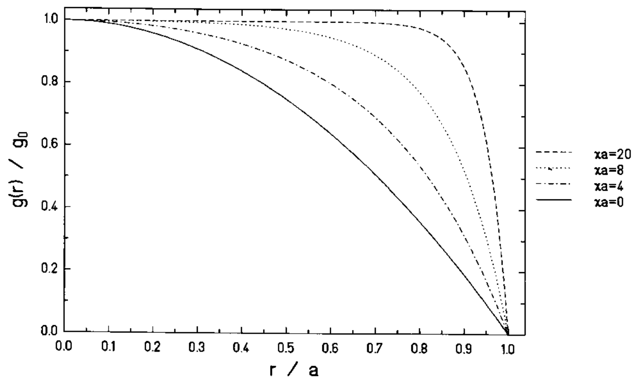
The constant g_0 represents the “limiting charge density” that the beam (almost) has on the beam axis. It is proportional to the tune squared!

8. Stationary phase-space distributions
8.3. WB distribution



Normalized effective potentials of self-consistent WB distributions.
The horizontal lines indicate the respective Debye lenghts.

8. Stationary phase-space distributions
8.3. WB distribution



Normalized line densities of self-consistent WB distributions for different numbers of the space-charge parameter κa.

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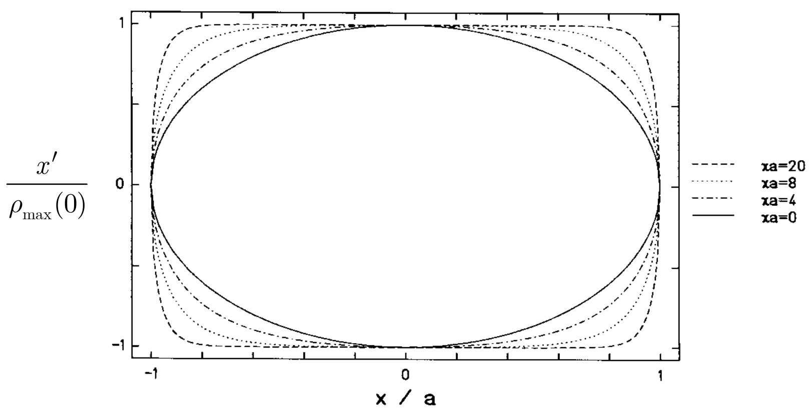
8. Stationary phase-space distributions
8.3. WB distribution



Normalized boundaries of the populated phase space of SC WB distributions for different numbers of the space-charge parameter κa.

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The definition $\kappa^2 = 4\pi^2 K c_0$ involves the as yet unknown normalization factor c_0. It is determined by the requirement

$$\frac{2\pi c\beta}{I}\int_{r=0}^{a} g(r)\, r\, dr \overset{!}{=} 1$$

Inserting the charge density $g_{\rm WB}(r)$ of the self-consistent WB distribution:

$$\frac{k_0^2 a^2}{K}\left(1 - \frac{2}{\kappa^2 a^2 I_0(\kappa a)}\int_0^{\kappa a} u I_0(u)\, du\right) \overset{!}{=} 1$$

Because of the identities

$$\int_0^{\kappa a} u I_0(u)\, du = \kappa a\, I_1(\kappa a), \qquad \frac{2 I_1(\kappa a)}{\kappa a} = I_0(\kappa a) - I_2(\kappa a)$$

we finally find the following *implicit equation* for κa:

$$\frac{K}{k_0^2 a^2} = \frac{I_2(\kappa a)}{I_0(\kappa a)}$$

Note: to solve this equation for κa we need the yet unknown beam edge a for the given RMS beam size!

The line density $n_{\rm WB}(r)$ is finally obtained from $g_{\rm WB}(r)$ as:

$$n_{\rm WB}(r) = \frac{c\beta}{I} g_{\rm WB}(r) = \frac{1}{\pi a^2}\frac{I_0(\kappa a) - I_0(\kappa r)}{I_2(\kappa a)} \underset{\kappa \to 0}{\longrightarrow} \frac{2}{\pi a^2}\left(1 - \frac{r^2}{a^2}\right)$$

This agrees with the "geometrical" WB!

We easily verify that $n_{\rm WB}(r)$ is normalized. With the normalization condition, the effective potential $V_{\rm eff,WB}(r)$ is now:

$$V_{\rm eff,WB}(r) - V_{\rm eff,WB}(0) = \frac{2k_0^2 a^2}{\kappa^2 a^2 I_0(\kappa a)}\left[I_0(\kappa r) - 1\right] = 2K\,\frac{I_0(\kappa r) - 1}{\kappa^2 a^2 I_2(\kappa a)}$$

We can identify $1/\kappa$ with the Debye length $\lambda_D = 1/\kappa$.

λ_D measures the *screening length* of the charge distribution. The charges arrange in a way to screen off the external field:

➔ inside the beam, the particles move (almost) force-free

➔ at the beam edge, the particles are reflected at a steep wall potential of the width λ_D.

As we know from Sec. 6.2, only the RMS values of different distributions can be compared ("equivalent beams").

➔ We obtain the desired beam parameters from the RMS envelope equation in terms of RMS values.

➔ As we do not know the actual beam radius *a*, we must express *a* in terms of the corresponding RMS value in the implicit equation for κa.

➔ We must calculate the ratios of "marginal" beam parameters to the corresponding RMS values, i.e. the ratios

$$\left.\frac{\overline{x^2}}{a^2}\right|_{\mathrm{WB}}, \quad \left.\frac{\overline{x'^2}}{\rho_{\max}^2(0)}\right|_{\mathrm{WB}}$$

Note that these ratios are *different* in general!

8. Stationary phase-space distributions
8.3. WB distribution

The second moment $\overline{x^2}$ is defined as

$$\overline{x^2} = \iint x^2 n(x,y)\,dx dy \overset{\text{round beam}}{=} \int_0^{2\pi} \cos^2\varphi\,d\varphi \int_0^a r^2 n(r)\,r dr = \pi \int_0^a r^3 n(r)\,dr$$

Inserting the line density $n_{\mathrm{WB}}(r)$ of the WB distribution, we get:

$$\left.\frac{\overline{x^2}}{a^2}\right|_{\mathrm{WB}} = \frac{1}{(\kappa a)^4} \int_0^{\kappa a} \frac{I_0(\kappa a) - I_0(\kappa r)}{I_2(\kappa a)} (\kappa r)^3\,d(\kappa r) = \frac{1}{2} + \frac{2}{(\kappa a)^2} - \frac{I_0(\kappa a)}{4 I_2(\kappa a)} \begin{cases} \xrightarrow[\kappa\to 0]{} \dfrac{1}{6} \\ \xrightarrow[\kappa\to\infty]{} \dfrac{1}{4} \end{cases}$$

We can now calculate κa for the given values RMS emittance $\tilde{\varepsilon}$, scaled current *K*, and zero current tune k_0 using the RMS envelope equation for a stationary round beam:

$$k_0^2 \overline{x^2} - \frac{K}{4} - \frac{\tilde{\varepsilon}^2}{\overline{x^2}} = 0$$

This is *not* an approximation since ε = const. for a stationary beam!

8. Stationary phase-space distributions
8.3. WB distribution

Combining

$$\frac{K}{k_0^2 a^2} = \frac{I_2(\kappa a)}{I_0(\kappa a)}, \quad \frac{\overline{x^2}}{a^2} = \frac{1}{2} + \frac{2}{(\kappa a)^2} - \frac{I_0(\kappa a)}{4 I_2(\kappa a)}, \quad k_0^2 \overline{x^2} - \frac{K}{4} - \frac{\tilde{\varepsilon}^2}{\overline{x^2}} = 0$$

yields an implicit equation for the dimensionless space charge parameter κa, given the distribution-independent quantities $k_0, K, \tilde{\varepsilon}$:

$$\frac{\dfrac{I_2(\kappa a)}{I_0(\kappa a)}}{\dfrac{1}{2} + \dfrac{2}{(\kappa a)^2} - \dfrac{I_0(\kappa a)}{4 I_2(\kappa a)}} = \frac{\dfrac{K}{k_0 \tilde{\varepsilon}}}{\dfrac{K}{8 k_0 \tilde{\varepsilon}} + \sqrt{1 + \left(\dfrac{K}{8 k_0 \tilde{\varepsilon}}\right)^2}}$$

We can solve this equation for κa by iteration ("regula falsi"). With κa, we can then calculate *spatial* extension a of the populated phase space:

$$\frac{K}{k_0^2 a^2} = \frac{I_2(\kappa a)}{I_0(\kappa a)} \quad \Rightarrow \quad a = \frac{\sqrt{K}}{k_0} \sqrt{\frac{I_0(\kappa a)}{I_2(\kappa a)}}$$

8. Stationary phase-space distributions
8.3. WB distribution

We finally need to calculate the ratio $\overline{x'^2} / \rho_{\max}^2(0)$. With the above expression for the maximum angle at r

$$\rho_{\max}^2(r) = 2[V_{\text{eff}}(a) - V_{\text{eff}}(r)] \quad \overset{\text{WB}}{\Rightarrow} \quad \rho_{\max}^2(r) = 4K \frac{I_0(\kappa a) - I_0(\kappa r)}{(\kappa a)^2 I_2(\kappa a)}$$

we have

$$\overline{x'^2} = \frac{\frac{1}{2} \int\limits_{r=0}^{a} \int\limits_{\rho^2=0}^{\rho_{\max}^2(r)} \rho^2 d(\rho^2) r dr}{\int\limits_{r=0}^{a} \int\limits_{\rho^2=0}^{\rho_{\max}^2(r)} d(\rho^2) r dr} = \frac{K}{(\kappa a)^2 I_2(\kappa a)} \frac{\int\limits_0^{\kappa a} [I_0(\kappa a) - I_0(\kappa r)]^2 \kappa r \, d(\kappa r)}{\int\limits_0^{\kappa a} [I_0(\kappa a) - I_0(\kappa r)] \kappa r \, d(\kappa r)}$$

$$= \frac{2K}{(\kappa a)^2 I_2(\kappa a)} \left[I_0(\kappa a) - \frac{I_1^2(\kappa a)}{2 I_2(\kappa a)} \right]$$

and hence the ratio

$$\left. \frac{\overline{x'^2}}{\rho_{\max}^2(0)} \right|_{\text{WB}} = \frac{1}{2} \frac{1}{I_0(\kappa a) - 1} \left[I_0(\kappa a) - \frac{I_1^2(\kappa a)}{2 I_2(\kappa a)} \right] \begin{cases} \xrightarrow[\kappa \to 0]{} \dfrac{1}{6} \\ \xrightarrow[\kappa \to \infty]{} \dfrac{1}{4} \end{cases}$$

$$\frac{K}{k_0}$$

8. Stationary phase-space distributions

8.4. Gaussian (GA) distribution (sketch)

with c_0 denoting a normalization constant and

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dV_{\text{eff,GA}}(r)}{dr}\right] = 2k_0^2 - 4\pi^2 K c_0 \int_{V_{\text{eff,GA}}(r)}^{\overline{H}_{\max}} \exp\left(-\frac{\overline{H}}{2\alpha^2\overline{H}_{\max}}\right) d\overline{H}$$

$$= 2k_0^2 - 4\pi^2 K c_0 \cdot 2\alpha^2 V_{\text{eff,GA}}(a)\left[\exp\left(-\frac{V_{\text{eff,GA}}(r)}{2\alpha^2 V_{\text{eff,GA}}(a)}\right) - \exp\left(-\frac{1}{2\alpha^2}\right)\right]$$

$$\exp(W(r)) = \exp\left(-\frac{V_{\text{eff,GA}}(r)}{2\alpha^2 V_{\text{eff,GA}}(a)}\right) - \exp\left(-\frac{1}{2\alpha^2}\right) - \frac{k_0^2}{\kappa_{\text{GA}}^2 \alpha^2 V_{\text{eff,GA}}(a)}, \quad \kappa_{\text{GA}}^2 = 4\pi^2 K c_0$$

which yields:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dW(r)}{dr}\right]=\kappa_{\mathrm{GA}}^{2}\exp\left(W(r)\right)$$

This equation has the analytical solution:

$$\exp\left(W(r)\right)=\frac{c_1}{r^2}\left(1-\frac{2}{\kappa_{\mathrm{GA}}^{2}}\right)\left(1+\tan\left\{\left(\tfrac{1}{2}c_1\kappa_{\mathrm{GA}}^{2}-1\right)\left(\ln r-c_2\right)^2\right\}\right)$$

The two integration constants, c_1 and c_2, are to be determined from the conditions that:

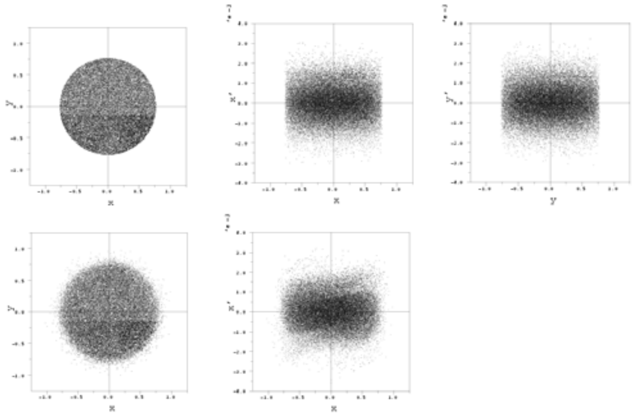
1. the charge density on the beam axis is equal to the limiting density g_0 (see Sec. 8.3),
2. the integral over line density $n_{\mathrm{GA}}(r)$ must be unity.

Definition of the SG distribution:

In the *high current regime*, we can approximate f_{GA} by the "semi-gaussian" (SG) distribution. It has the following properties:

1. uniform density in real space with sharp edge ($\lambda_{\mathrm{D}} \rightarrow 0$)
2. Gaussian density (truncated) in velocity space
3. rectangular boundary in the x,x' and y,y' phase space projections.

➔The SG distribution is the self-consistent GA distribution in the limit of infinite current density.



"Semi-Gaussian (SG) distribution in a continuous focusing channel at $\sigma_0 = 60^o$, $\sigma = 15^o$. Upper row: initial state, lower row: after one "period".

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JOHANN WOLFGANG GOETHE UNIVERSITÄT FRANKFURT AM MAIN

8. Stationary phase-space distributions

8.5. “Stationary” beams under periodic focusing

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Recall the definition of *equivalent focusing channels*: Two focusing channels are referred to as *equivalent* if both the period lengths S and the zero current tunes σ_0 agree.

We RMS match a self-consistent WB distribution that was generated for a *continuous* focusing channel to

1. a periodic solenoid channel
2. a periodic quadrupole channel

by means of the linear, area-preserving matching transformation:

$$\begin{pmatrix}\hat{x}\\ \hat{x}'\end{pmatrix} = \begin{pmatrix}\sqrt{\hat{\beta}_x/\beta_x} & 0\\ -\dfrac{\hat{\alpha}_x}{\sqrt{\hat{\beta}_x\beta_x}} & \sqrt{\beta_x/\hat{\beta}_x}\end{pmatrix}\begin{pmatrix}x\\ x'\end{pmatrix}, \quad \begin{pmatrix}\hat{y}\\ \hat{y}'\end{pmatrix} = \begin{pmatrix}\sqrt{\hat{\beta}_y/\beta_y} & 0\\ -\dfrac{\hat{\alpha}_y}{\sqrt{\hat{\beta}_y\beta_y}} & \sqrt{\beta_y/\hat{\beta}_y}\end{pmatrix}\begin{pmatrix}y\\ y'\end{pmatrix}$$

Question: How does the matching transformation transform the RMS emittances ε_x, ε_y, and their respective s-derivatives?

We easily convince ourselves by insertion that

$$\hat{\varepsilon}_x^2 = \overline{\hat{x}^2}\,\overline{\hat{x}'^2} - \overline{\hat{x}\hat{x}'}^2 = \varepsilon_x^2, \qquad \hat{\varepsilon}_y^2 = \varepsilon_y^2$$

in agreement with the fact that the matrices of the matching transformation have unit determinants.

➔ the matching transformation preserves the RMS emittances.

In order to determine the transformation of the s-derivatives of ε_x, ε_y, we need to know the mapping of the electric fields E_x, E_y under the above matching transformation.

We recall that for an unbunched beam of *elliptic symmetry*, we have

$$\overline{xE_x} = \frac{I}{4\pi\varepsilon_0 c\beta}\,\frac{\sqrt{\overline{x^2}}}{\sqrt{\overline{x^2}}+\sqrt{\overline{y^2}}}, \qquad \overline{yE_y} = \frac{I}{4\pi\varepsilon_0 c\beta}\,\frac{\sqrt{\overline{y^2}}}{\sqrt{\overline{x^2}}+\sqrt{\overline{y^2}}}$$

➔ if we apply a *symmetric* matching transformation of the real space coordinates

$$\hat{x} = f_1\,x, \quad \hat{y} = f_1\,y, \quad f_1 = \sqrt{\hat{\beta}_x / \beta_x} = \sqrt{\hat{\beta}_y / \beta_y}$$

then

$$\overline{\hat{x}\hat{E}_x} = \overline{xE_x}, \quad \overline{\hat{y}\hat{E}_y} = \overline{yE_y}$$

and hence

$$\hat{E}_x = \frac{1}{f_1}E_x, \quad \hat{E}_y = \frac{1}{f_1}E_y$$

For an unbunched beam, the electric fields E_x, E_y scale inversely to the real space coordinates x, y.

With
$$\hat{x}' = \frac{1}{f_1} x' - \frac{\hat{\alpha}_x}{\hat{\beta}_x} f_1 \, x, \qquad \hat{y}' = \frac{1}{f_1} y' - \frac{\hat{\alpha}_y}{\hat{\beta}_y} f_1 \, y$$

we can now calculate the s-derivatives of the RMS emittances in the transformed system

$$\begin{aligned}
\frac{d}{ds}\hat{\varepsilon}_x^2\Big|_{s=0} &= \frac{2e}{mc^2\beta^2\gamma^3}\left(\overline{\hat{x}^2}\;\overline{\hat{x}'\hat{E}_x} - \overline{\hat{x}\hat{x}'}\;\overline{\hat{x}\hat{E}_x}\right) \\
&= \frac{2e}{mc^2\beta^2\gamma^3}\left[\overline{x^2}\;\overline{x'E_x} - \frac{\hat{\alpha}_x}{\hat{\beta}_x} f_1^2\,\overline{x^2}\;\overline{xE_x} - \left(\overline{xx'}\;\overline{xE_x} - \frac{\hat{\alpha}_x}{\hat{\beta}_x} f_1^2\,\overline{x^2}\;\overline{xE_x}\right)\right] \\
&= \frac{2e}{mc^2\beta^2\gamma^3}\left(\overline{x^2}\;\overline{x'E_x} - \overline{xx'}\;\overline{xE_x}\right) = \frac{d}{ds}\varepsilon_x^2\Big|_{s=0}
\end{aligned}$$

➔ a symmetric matching transformation also preserves the s-derivatives of the RMS emittances.

➔ We have *no initial emittance growth* if we RMS match a beam that is self-consistent within a lattice with *continuous* focusing forces to a *periodic* lattice.

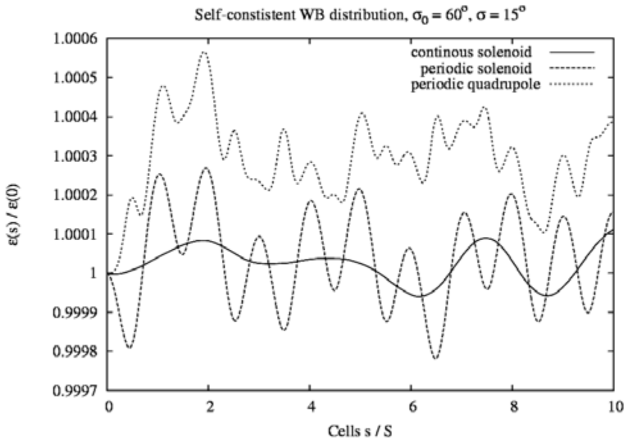
Periodic focusing:
$$\frac{\partial f}{\partial s} \neq 0, \qquad \overline{x^2}, \overline{y^2} \neq \text{ const.}$$

The location of a particle after δs depends on its angle x':
$$x(s + \delta s) = x(s) + x'(s)\,\delta s$$

➔ For non-KV distributions, the real-space and the velocity-space probability densities are different.

➔ A change of the beam size is *always* accompanied by a change of the charge density profile, and hence by a change of the RMS emittance.

8. Stationary phase-space distributions
8.5. Periodic focusing



The *S*-periodic part of the RMS emittance oscillation is due to the "breathing" of the beam envelope under periodic focusing forces. For non-KV distributions, a change of the beam width causes a change of the free (nonlinear) field energy – and hence an emittance change.

Emittance growth factors versus cell number for "self-consistent" WB distributions under different external focusing schemes.

8. Stationary phase-space distributions
8.5. Periodic focusing

We conclude: the procedure to

1. generate a self-consistent distribution for a continuous focusing channel that is equivalent to the given periodic channel, and
2. to RMS match the generated distribution to the periodic channel

is the best (known) method to obtain (almost) self-consistent initial distributions for periodic focusing channels.