Einstein Year 2005:

The Lorentz Transformation as a Canonical Transformation in the Extended Phase Space

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Basics

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- The transformation rule between inertial systems is given by the Lorentz transformation.
- The correlation between Hamiltonian systems is *always* established by a *canonical transformation*.
- $\rightsquigarrow$ The Lorentz transformation *must* constitute a particular canonical transformation.
- Problem: a conventional canonical transformation (CT) preserves time, i.e. the time t is *always* the common independent variable of both the original and the destination system.

Basics (continued)

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  are made possible.
- Moreover, the relation between non-Lorentz-invariant and Lorentz-invariant Hamiltonians will become transparent, e.g.,

$$H_{\mathrm{NL}}(p) = \frac{p^2}{2m} + mc^2 \quad \Longleftrightarrow \quad H_{\mathrm{L}}(p) = \sqrt{p^2c^2 + m^2c^4}$$

Outline

- Principle of least action and its general formulation
- Extended set of canonical equations
- Generalized theory of canonical transformations
- Example: The Lorentz transformation
- Conclusions

Principle of least action

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With time t the independent variable, a *path* γ in the $2n$ dimensional phase space is defined by

$$\gamma : \left\{ \left(\vec{q}, \vec{p}\right) \in \mathbb{R}^{2n} \,\middle|\, \vec{q} = \vec{q}(t), \vec{p} = \vec{p}(t), t_0 \leq t \leq t_1 \right\} .$$

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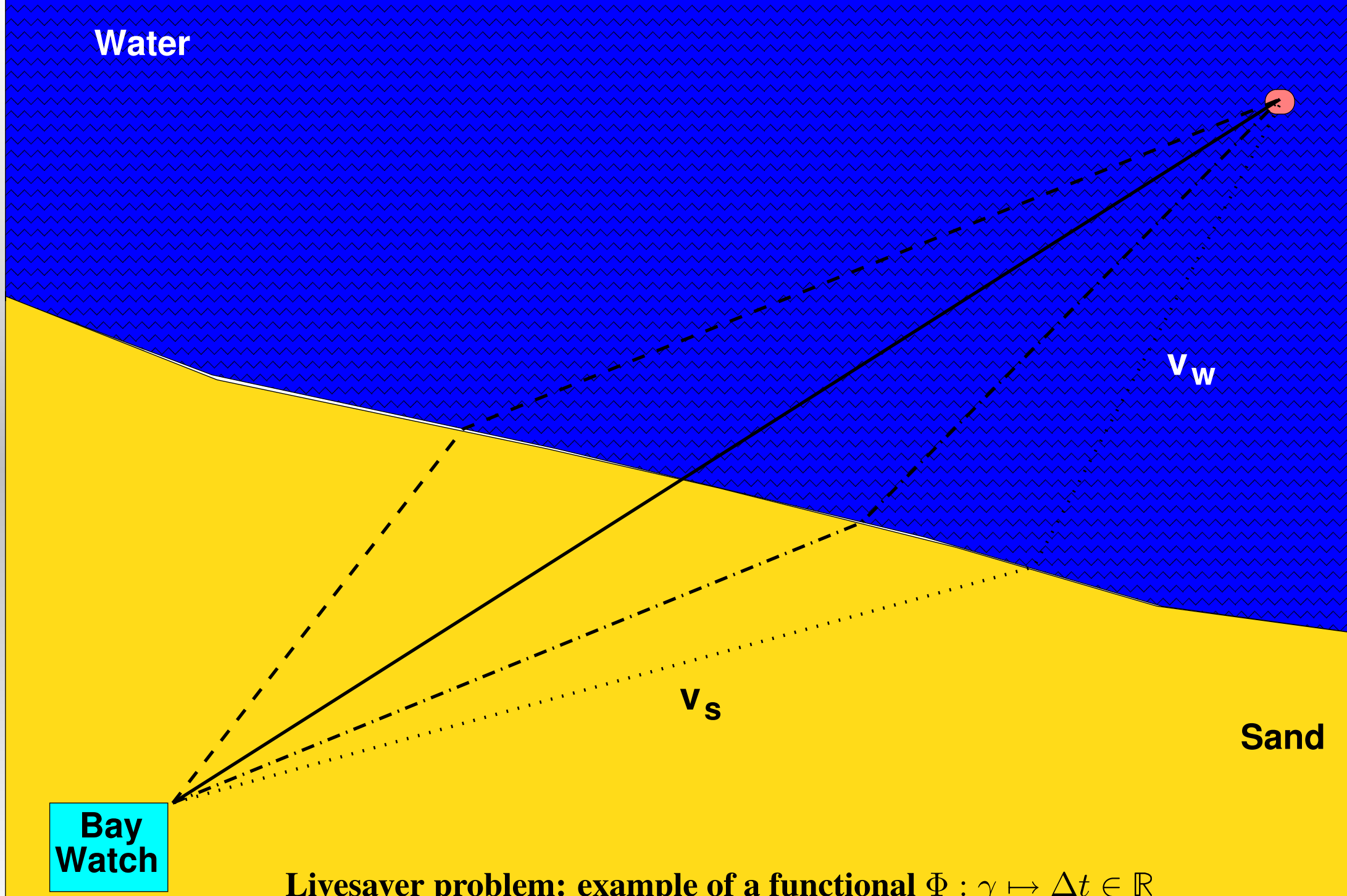
We formulate the principle of least action via a *functional* Φ, hence with a mapping of the set of paths γ into $\mathbb{R}$

$$\Phi(\gamma) = \int_{t_0}^{t_1} \left[\vec{p}(t) \frac{d\vec{q}(t)}{dt} - H\left(\vec{q}(t), \vec{p}(t), t\right) \right] dt \, .$$

The function $H : \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$ denotes the Hamiltonian.



Livesaver problem: example of a functional $\Phi : \gamma \mapsto \Delta t \in \mathbb{R}$

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Calculus of variations:

The functional $\Phi(\gamma)$ takes on an extreme value ($\delta\Phi(\gamma_{\mathrm{ext}}) = 0$), exactly if the phase-space path $\gamma_{\mathrm{ext}} : \big(\vec{q}(t), \vec{p}(t)\big)$ satisfies the "canonical equations"

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\,, \qquad i = 1, \ldots, n\,.$$

Let us look back to the variational problem $\delta\Phi(\gamma) \stackrel{!}{=} 0$

$$\delta\Phi(\gamma) = \delta \int_{t_0}^{t_1} \left[\vec{p}(t) \frac{d\vec{q}(t)}{dt} - H\big(\vec{q}(t), \vec{p}(t), t\big) \right] dt \stackrel{!}{=} 0\,.$$

We observe: the time t plays a *twofold* role, namely that of the formal *integration variable* and that of an *external parameter* in the argument list of the Hamiltonian.

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Calculating the variation $\delta\Phi(\gamma)$, the time t is *not* varied.

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$\rightsquigarrow$ We must *separate* the explicit t-dependence of the Hamiltonian from the formal integration variable.

A more general form of the variational problem is obtained with the new integration variable s by substituting $t = t(s)$,

$$\delta \int_{s_0}^{s_1} \left[\vec{p}(s) \, \frac{d\vec{q}(s)}{ds} - H\big(\vec{q}(s), \vec{p}(s), t(s)\big) \frac{dt(s)}{ds} \right] ds \stackrel{!}{=} 0 \, .$$

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The symmetric form of the integrand suggests to define the $2n+2$ dimensional "extended phase space" by introducing

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$\rightsquigarrow$ $h = h(s) \in \mathbb{R}$ must be understood as the *value* of the Hamiltonian $H(\vec{q}, \vec{p}, t)$, hence as the system's "instantaneous energy"

$$h(s) \stackrel{\not\equiv}{=} H\big(\vec{q}(s), \vec{p}(s), t(s)\big) \, .$$

Defining the extended vectors $\vec{q}_1 = (\vec{q}, t)$ and $\vec{p}_1 = (\vec{p}, -h)$, the variational integral can be converted into the standard form

$$\delta \int_{s_0}^{s_1} \left[\vec{p}_1(s) \, \frac{d\vec{q}_1(s)}{ds} - H_1\big(\vec{q}_1(s), \vec{p}_1(s)\big) \right] ds \stackrel{!}{=} 0 \,,$$

with the extended Hamiltonian H_1 as the *implicit function*

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<u>We observe:</u> with H_1, we encounter the *extended* functional exactly in the form of the *conventional* functional.

$\rightsquigarrow$ The variation of the functional vanishes again if the *extended phase-space path* $\big(\vec{q}_1(s), \vec{p}_1(s)\big)$ satisfies the *extended set of canonical equations*,

$$\frac{dq_i}{ds} = \frac{\partial H_1}{\partial p_i} \,, \qquad \frac{dp_i}{ds} = -\frac{\partial H_1}{\partial q_i} \,, \qquad i = 1, \ldots, n+1 \,.$$

In terms of the quantities $\vec{q}$, $\vec{p}$, t, h and H, this means

$$\frac{d\vec{q}}{ds} = \frac{\partial H_1}{\partial \vec{p}} = \frac{dt}{ds}\frac{\partial H}{\partial \vec{p}} \quad , \quad \frac{d\vec{p}}{ds} = -\frac{\partial H_1}{\partial \vec{q}} = -\frac{dt}{ds}\frac{\partial H}{\partial \vec{q}} \, ,$$

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- $\rightsquigarrow$ The principle of least action is equally satisfied for *all* differentiable parameterizations of time $t = t(s)$.
- Exactly this freedom allows us to define more general canonical transformations in the extended phase space.

Canonical transformations

General condition for canonical transformations:

The variational principle must be maintained.

This means in the *conventional* description

$$\delta \int_{t_0}^{t_1} \left[\vec{p}\,\dot{\vec{q}} - H\left(\vec{q}, \vec{p}, t\right) \right] \, dt = \delta \int_{t_0}^{t_1} \left[\vec{p}\,'\dot{\vec{q}}\,' - H'\left(\vec{q}\,', \vec{p}\,', t\right) \right] \, dt \, .$$

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$\rightsquigarrow$ Canonical transformations that correlate two systems on the basis of *different time scales* t, t' are *not* possible.

$\rightsquigarrow$ Only a CT in the extended phase space can do that job

$$H(\vec{q},\vec{p},t) \xrightarrow{\text{CT in the extended phase space}} H'(\vec{q}\,',\vec{p}\,',t')\,.$$

According to the generalized form of the variational principle, the condition for transformations to be canonical writes in the *extended phase-space description*

$$\delta \int_{s_1}^{s_2} \left[\vec{p}_1 \frac{d\vec{q}_1}{ds} - H_1\left(\vec{q}_1, \vec{p}_1\right) \right] ds = \delta \int_{s_1}^{s_2} \left[\vec{p}_1{}' \frac{d\vec{q}_1{}'}{ds} - H_1'\left(\vec{q}_1{}', \vec{p}_1{}'\right) \right] ds \, .$$

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$\rightsquigarrow$ The integrands may differ by the total derivative dF_1/ds of a function $F_1(\vec{q}_1, \vec{q}_1{}', s)$

$$\frac{dF_1}{ds} = \vec{p}_1 \frac{d\vec{q}_1}{ds} - H_1 - \vec{p}_1{}' \frac{d\vec{q}_1{}'}{ds} + H_1' \, .$$

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Because of

$$\frac{dF_1}{ds} = \frac{\partial F_1}{\partial \vec{q}_1} \frac{d\vec{q}_1}{ds} + \frac{\partial F_1}{\partial \vec{q}_1{}'} \frac{d\vec{q}_1{}'}{ds} + \frac{\partial F_1}{\partial s} \,,$$

we find the transformation rules by comparing the coefficients

$$\vec{p}_1 = \frac{\partial F_1}{\partial \vec{q}_1}\,, \qquad \vec{p}_1{}' = -\frac{\partial F_1}{\partial \vec{q}_1{}'}\,, \qquad H_1' = H_1 + \frac{\partial F_1}{\partial s}\,.$$

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The leftmost two rules written in terms of the quantities $\vec{q}$, $\vec{p}$, t and h, respectively (remember: $p_{n+1} \equiv -h, q_{n+1} \equiv t$):

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Accordingly, $F_1(\vec{q}_1, \vec{q}_1{}', s) \equiv F_1(\vec{q}, t, \vec{q}\,', t', s)$ is the *generating function* of the extended canonical transformation.

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By means of a *Legendre* transformation

$$F_2\big(\vec{q}_1, \vec{p}_1{}', s\big) = F_1\big(\vec{q}_1, \vec{q}_1{}', s\big) + \vec{q}_1{}'\,\vec{p}_1{}'\,,$$

the generating function F_1 can be converted into a generating function of type F_2.

The transformation rules associated with $F_2(\vec{q}, \vec{p}', t, h', s)$ are

$$\vec{p} = \frac{\partial F_2}{\partial \vec{q}}\,,\ \vec{q}\,' = \frac{\partial F_2}{\partial \vec{p}\,'}\,,\ h = -\frac{\partial F_2}{\partial t}\,,\ t' = -\frac{\partial F_2}{\partial h'}\,,\ H_1' = H_1 + \frac{\partial F_2}{\partial s}\,.$$

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According to the definition of H_1, the transformation rule for the conventional (non-extended) Hamiltonian H writes

$$\big(H' - h'\big)\frac{dt'}{ds} = \big(H - h\big)\frac{dt}{ds} + \frac{\partial F}{\partial s}\,.$$

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If the generating function F is not explicitly s-dependent, we can eliminate the evolution parameter s

$$\big(H' - h'\big)\frac{\partial t'}{\partial t} = H - h\,.$$

The transformation rules associated with $F_2(\vec{q}, \vec{p}', t, h', s)$ are

$$\vec{p} = \frac{\partial F_2}{\partial \vec{q}}\,, \quad \vec{q}\,' = \frac{\partial F_2}{\partial \vec{p}\,'}\,, \quad h = -\frac{\partial F_2}{\partial t}\,, \quad t' = -\frac{\partial F_2}{\partial h'}\,, \quad H_1' = H_1 + \frac{\partial F_2}{\partial s}\,.$$

According to the definition of H_1, the transformation rule for the conventional (non-extended) Hamiltonian H writes

$$\big(H' - h'\big)\frac{dt'}{ds} = \big(H - h\big)\frac{dt}{ds} + \frac{\partial F}{\partial s}\,.$$

If the generating function F is not explicitly s-dependent, we can eliminate the evolution parameter s

$$\big(H' - h'\big)\frac{\partial t'}{\partial t} = H - h\,.$$

$\rightsquigarrow$ As we will see in the next slide, these relations *generalize* the conventional transformation rule for the Hamiltonians H' and H.

With $f_2(\vec{q}, \vec{p}', t)$ a conventional generating function, we may define its extension $F_2(\vec{q}, \vec{p}', t, h')$ by

$$F_2(\vec{q}, \vec{p}', t, h') = f_2(\vec{q}, \vec{p}', t) - t\, h' \,.$$

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For this particular F_2, the transformation rules are obtained as

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$\rightsquigarrow$ The extended CT rules allow more general relations of $H \leftrightarrow H'$ and $t \leftrightarrow t'$ than the conventional CT rules.

Example: Lorentz transformation

We consider two Cartesian frames of reference (x, y, z) and (x', y, z) that move with respect to each other in x-direction at a constant velocity $c\beta$, i.e. at $\gamma = 1/\sqrt{1-\beta^2} \geq 1$.

In the extended phase space, a generating function F_2 exists that exactly yields the Lorentz transformation rules

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namely

$$\begin{aligned} p_x &= \frac{\partial F_2}{\partial x} = \gamma\left(p'_x + \beta\frac{h'}{c}\right), \qquad & h &= -\frac{\partial F_2}{\partial t} = \gamma\,(h' + \beta c\, p'_x)\,, \\ x' &= \frac{\partial F_2}{\partial p'_x} = \gamma\,(x - \beta c\, t)\,, \qquad & t' &= -\frac{\partial F_2}{\partial h'} = \gamma\left(t - \frac{\beta}{c}x\right). \end{aligned}$$

With the real angle $\alpha = \operatorname{arcosh}\gamma = \operatorname{arsinh}\beta\gamma$, the transformation can be written as the imaginary rotation

$$\begin{pmatrix} x' \\ ict' \\ p'_x \\ ih'/c \end{pmatrix} = \begin{pmatrix} \cos i\alpha & \sin i\alpha & 0 & 0 \\ -\sin i\alpha & \cos i\alpha & 0 & 0 \\ 0 & 0 & \cos i\alpha & \sin i\alpha \\ 0 & 0 & -\sin i\alpha & \cos i\alpha \end{pmatrix} \begin{pmatrix} x \\ ict \\ p_x \\ ih/c \end{pmatrix}$$

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$\rightsquigarrow$ The "Minkowski metric" is a conserved quantity under Lorentz transformations

$$x'^2 - c^2 t'^2 = x^2 - c^2 t^2 \,, \qquad p_x'^2 - h'^2/c^2 = p_x^2 - h^2/c^2 \,.$$

Because of $\partial F_2/\partial s = 0$, we have $H_1' = H_1$. With $\partial t'/\partial t = \gamma$, the transformation rule for the conventional Hamiltonian H is obtained as

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Together with the coordinate transformation rule

$$h = \gamma h' + \beta\gamma c\, p_x',$$

we finally find the general rule for a one-particle system

$$H(p_x, x, t) = \gamma H'(p_x', x', t') + \beta\gamma c p_x'\,.$$

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$\rightsquigarrow$ Einstein's principle of relativity requires any one-particle Hamiltonian H to be form-invariant under this relation.

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- Only *one* particle or *one* field can be consistently transformed.
- For the relativistic consistent description of multi-particle systems (e.g. nuclear reactions that involve several particles), a higher dimensional phase-space approach is necessary.

We consider the conventional Hamiltonian of a single free particle of rest mass m

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In the *extended phase space*, the corresponding Lorentz-invariant Hamiltonian of a single free particle looks like

$$H_{\mathrm{L}}(p,h) = \frac{1}{2m}\left[p^2 - \frac{\left(h - mc^2\right)^2}{c^2}\right] + mc^2 \,.$$

We project the extended phase-space representation of H_{L} into the *conventional* one by replacing the *value* h of the Hamiltonian H_{L} by the Hamiltonian itself

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Solving this equation for H_{L}, we find the familiar result of the Lorentz-invariant Hamiltonian of a single free particle

$$H_{\mathrm{L}}(p) = \sqrt{p^2c^2 + m^2c^4}\,.$$

In order to prove the form-invariance of $H = \sqrt{p^2c^2 + m^2c^4}$, we make use of the transformation rule for the Hamiltonians

$$H(p, x, t) = \gamma H'(p', x', t') + \beta\gamma\, p'c$$

and the coordinate transformation rule

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Because of $\gamma^2(1 - \beta^2) \equiv 1$, this yields the expected result

$$H'(p') = \sqrt{p'^2c^2 + m^2c^4} \,.$$

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- The generalized canonical transformation theory allows us to formulate transformations that also map the *time-scales* of original and destination systems.
- The generalized theory thus permits the formulation of the Lorentz transformation as an extended CT.
- In the extended phase-space representation, a Lorentz-invariant Hamiltonian is easily constructed.
- A usual Lorentz-invariant Hamiltonian is then given by its *projection* into the *conventional phase space*.

Publication

- J. Struckmeier, *Hamiltonian dynamics on the symplectic extended phase space for autonomous and non-autonomous systems*,
  J. Phys. A: Math. Gen. **38** (2005) 1257–1278
- This talk may be downloaded from
  “http://www.gsi.de/˜struck”